

The validity of posterior expansions based on Laplace's method (Geisser et al., 1990)

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Laplace's method is a classical tool for approximating high-frequency integrals

$$\int_{\mathbb{R}} g(x) e^{-nh(x)} dx = \sqrt{\frac{2\pi}{nh''(x_0)}} e^{-nh(x_0)} \left(g(x_0) + O(n^{-1}) \right)$$

as $n \rightarrow \infty$, where x_0 is the unique global minimizer of h . The key idea is to approximate

$$h(x_0) + \frac{1}{2}(h''(x_0) - \epsilon)(x - x_0)^2 \leq h(x) \leq h(x_0) + \frac{1}{2}(h''(x_0) + \epsilon)(x - x_0)^2$$

for sufficiently small $0 < \epsilon < h''(x_0)$ around a δ -neighborhood of x_0 . This essentially turns the integrand of interest into an unnormalized Gaussian density, with nearly all of its mass concentrated in a vanishingly small neighborhood of x_0 !

In Bayesian inference, the posterior density of θ under some prior $\pi(\theta)$ and likelihood $L(y | \theta)$ is described by

$$\mathbb{E}[g(\theta) | y] = \frac{\int g(\theta) L(y | \theta) \pi(\theta) d\theta}{\int L(y | \theta) \pi(\theta) d\theta},$$

which can be approximated using Laplace's method, expanding around the MAP $\hat{\theta}$. The main difference between this and the classical setup is that the log-likelihood function h (hence its maximizer x_0 too) depends on the data $y_1, \dots, y_n$. Geisser et al. [1990] formalizes conditions under which this kind of approximation does indeed work; call this *Laplace regularity* of log-likelihoods. One (very neat!) consequence is that for nice enough likelihoods, we have posterior concentration inequalities of the form

$$\mathbb{P}(|\theta - \hat{\theta}| \geq \delta | y) < e^{-nc}.$$

References

S Geisser, J Hodges, S Press, and A ZeUner. The validity of posterior expansions based on laplace's method. *Bayesian and likelihood methods in statistics and econometrics*, 7:473, 1990.