PH2103: Physics Laboratory III

Polarization of light

Polarisers and waveplates, Malus Law.

Satvik Saha*

19MS154

Indian Institute of Science Education and Research, Kolkata,
Mohanpur, West Bengal, 741246, India.

November 17, 2020

Abstract

In this experiment, we verify the Malus Law of polarization and study the effect of polarisers and waveplates on the intensity of light.

1 Theory

We have seen in our discussion of Lissajous figures that the superposition of two orthogonal vectors, whose amplitudes vary sinusoidally with the same frequency but with some phase difference, produces a new vector whose tip traces an ellipse. In other words, consider the parametric curve described by

$$x(t) = A\cos(\omega t), \qquad y(t) = B\cos(\omega t + \phi).$$

Eliminating the parameter t gives

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2xy}{AB}\cos\phi = \sin^2\phi.$$

Conversely, any light wave polarized as such (elliptically, which includes circular and linear[1]) can be decomposed into two sinusoidal components along any two perpendicular axes. A *polariser* functions by removing the component along one of its axes and allowing the remainder to pass. A *waveplate* functions by introducing an additional phase difference between the components along its axes (informally called the fast and slow axes).

Polarisers Consider a polariser whose transmission axis is inclined at an angle θ with respect to the oscillation of a linearly polarised wave. If this wave is of the form $E(t) = E_0\cos\omega t$, note that its intensity is given by

$$I_0 = \epsilon_0 c\langle E^2\rangle = \frac{1}{2}\epsilon_0 c E_0^2.$$

At any moment, the electric field vector can be decomposed into components along the transmission axis, and perpendicular to the transmission axis.

$$E_{\|}(t) = E_0\cos\omega t\cos\theta, \qquad E_{\perp}(t) = E_0\cos\omega t\sin\theta.$$

*Email: ss19ms154@iiserkol.ac.in

[1] A line is simply a degenerate ellipse

When passing through the polariser, only $E_{\|}$ survives. Its intensity is simply

$$I = \epsilon_0 c \langle E_{\|}^2 \rangle = \frac{1}{2}\epsilon_0 c E_0^2 \cos^2\theta.$$

Thus, we have obtained a relation between the initial and final intensities. This is called the Malus Law.

$$I = I_0 \cos^2\theta.$$

Of course, this holds only for linearly polarised light. If instead we start with circularly polarized light, the components along and perpendicular to the transmission axis will always have the same amplitude, regardless of the orientation of the polariser[2] (this is a simple consequence of symmetry). Thus, the component passing through must have exactly half the intensity of the original light, so $I = I_0/2$.

An analogous argument can be used for unpolarised light. Different waves in the incoming beam have their electric fields oscillating in different, random orientations. Thus, we can approximate θ as a random variable which varies uniformly over $[0, 2\pi]$. The average of $\cos^2\theta$ is thus $\langle \cos^2\theta \rangle = 1/2$, so the transmitted intensities of all these waves adds up to $I_0/2$, half of the incoming intensity.

Waveplates A waveplate comprises of an anisotropic medium which exhibits different refractive indices along its fast and slow axes. If light passes perpendicular to both these axes over a thickness t, the components along the fast and slow axes accumulate a path difference $d = t\Delta n$, where Δn is the difference in the fast and slow refractive indices. This corresponds to a phase difference of $\delta = 2\pi t \Delta n/\lambda$.

Suppose a waveplate imparts a phase difference of δ between components along its fast and slow axis. Thus, a linearly polarised wave striking the waveplate at an angle θ has components as described earlier. We add a phase δ to one of them to obtain

$$E_{\|}(t) = E_0 \cos(\omega t)\cos\theta, \qquad E_{\perp}(t) = E_0 \cos(\omega t + \delta)\sin\theta.$$

Thus, the waveplate changes the nature of the elliptic polarisation. Note that the intensity of the outgoing beam, which is equal to the sum of intensities of both outgoing components, remains unaltered.

$$I = I_{\|} + I_{\perp} = \frac{1}{2}\epsilon_0 c (E_0 \cos^2\theta + E_0 \sin^2\theta) = \frac{1}{2}\epsilon_0 c E_0^2 = I_0.$$

In the special case that $\theta = \pi/4$ and $\delta = \pi/2$, note that

$$E_{\|}(t) = \frac{1}{\sqrt{2}} E_0 \cos(\omega t), \qquad E_{\perp}(t) = -\frac{1}{\sqrt{2}} E_0 \sin(\omega t).$$

Thus, we have converted linearly polarised light into circularly polarised light. We can clearly see that $I_{\|} = I_{\perp} = I_0/2$. The principle of reversibility shows that we can also convert circularly polarised light into linearly polarised light by running this system backwards. This particular waveplate is called a *quarter waveplate*, because it imparts a path difference of $\lambda/4$.

In the special case that $\delta = \pi$, note that

$$E_{\|}(t) = E_0 \cos(\omega t)\cos\theta, \qquad E_{\perp}(t) = -E_0 \cos(\omega t)\sin\theta.$$

Thus, the linearly polarized light remains linear, but has been reflected about the parallel component, i.e. deflected by an angle 2θ. This particular waveplate is called a *half waveplate*,

[2] In all these cases, we of course assume that the polariser/waveplate is normal to the direction of propagation of the waves.

because it imparts a path difference of $\lambda/2$.

In any case, note that when the incoming linearly polarised beam is directed along one of the axes, i.e. one of $E_\parallel$ and $E_\perp$ is zero, the waveplate has no effect. Otherwise, the waveplate has the effect of distributing the intensity of the beam over both axes. This can be checked by removing one of the components using a polariser. Additionally, a polariser has no effect on a linearly polarised beam only when their axes are aligned too (θ= 0 in the Malus Law). Thus, such a waveplate-polariser system will produce a full intensity outgoing beam only when all of their axes (oscillation of the electric field, fast/slow axis of the waveplate, transmission axis of the polariser) are aligned.

Imperfect polarisers Suppose that the polarisers used are imperfect, such that the transmission axis allows a fraction $\beta \approx 1$ of the electric field amplitude to pass and the perpendicular axis allows a fraction $\alpha \approx 0$ of the electric field amplitude. Linearly polarised light striking this polariser such that its polarisation angle is inclined by θ with the transmission axis will thus transform into

$$E_\parallel = \beta E_0 \cos(\omega t)\cos\theta, \qquad E_\perp = \alpha E_0 \cos(\omega t)\sin\theta.$$

The intensity distribution is of the form

$$I(\theta) = \frac{1}{2}\epsilon_0 E_0^2(\beta^2\cos^2\theta + \alpha^2\sin^2\theta) = I_0(\alpha^2 + (\beta^2 - \alpha^2)\cos^2\theta).$$

Note that $I_{max} = \beta^2 I_0$ and $I_{min} = \alpha^2 I_0$. The quantity

$$\mathcal{P} = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{\beta^2 - \alpha^2}{\beta^2 + \alpha^2}$$

is called the degree of polarisation. Comparing the distribution with the curve $a + b\cos^2(x - c)$, we note that $I_0\alpha^2 = a$ and $I_0(\beta^2 - \alpha^2) = b$, so $\mathcal{P} = b/(2a + b)$.

With a slightly different definition of the degree of polarisation, we may write

$$\mathcal{P}' = 1 - \frac{I_{min}}{I_{max}} = 1 - \frac{\alpha^2}{\beta^2} = \frac{b}{a + b}.$$

Suppose instead that we start with elliptically polarised light of the form $E_x = A\cos(\omega t)$, $E_y = B\sin(\omega t)$[3]. We see that

$$I_0 = \frac{1}{2}\epsilon_0 c(A^2 + B^2) = \frac{1}{2}\epsilon_0 c E_0^2.$$

Rotating the frame by θ and scaling by β and α, we see that

$$E_\parallel = \beta\left(A\cos\theta\cos(\omega t) - B\sin\theta\sin(\omega t)\right), \qquad E_\perp = \alpha\left(A\sin\theta\cos(\omega t) + B\cos\theta\sin(\omega t)\right).$$

The intensity distribution is thus of the form

$$I(\theta) = \epsilon_0 c\left\langle \beta^2(A\cos\theta\cos(\omega t) - B\sin\theta\sin(\omega t))^2 + \alpha^2(A\sin\theta\cos(\omega t) + B\cos\theta\sin(\omega t))^2\right\rangle.$$

Note that

$$\langle (A\cos\theta\cos(\omega t) - B\sin\theta\sin(\omega t))^2\rangle = \frac{1}{2}A^2\cos^2\theta + \frac{1}{2}B^2\sin^2\theta,$$
$$\langle (A\sin\theta\cos(\omega t) + B\cos\theta\sin(\omega t))^2\rangle = \frac{1}{2}A^2\sin^2\theta + \frac{1}{2}B^2\cos^2\theta.$$

[3]It is always possible to choose appropriate axes to resolve the components this way. Specifically, any ellipse has this parametrization along the major and minor axes.

This is because the cross terms $\langle\cos(\omega t)\sin(\omega t)\rangle = 0$. Thus,

$$I(\theta) = \frac{1}{2}\epsilon_0 c\left[(\beta^2A^2 + \alpha^2B^2)\cos^2\theta + (\beta^2B^2 + \alpha^2A^2)\sin^2\theta\right].$$

With an appropriate choice of constants, this again simplifies to the form

$$I(\theta) = a + b\cos^2\theta.$$

2 Experimental setup

Light from a diode laser is passed through a polariser, which is oriented so as to obtain linearly polarised light of the maximum possible intensity. Once this adjustment is made, the light is passed through another polariser (an analyser). The analyser is rotated and the resultant intensity is measured as a function of this rotation angle.

Now, rotate the analyser so that the transmitted intensity is minimum. This must be the crossed position, where the transmission axes of the polariser and analyser are perpendicular. Insert a quarter waveplate between them, and rotate it such that the transmitted intensity is minimum once again. In this orientation, for the waveplate to have had no effect, we deduce that the waveplate anisotropy axes are aligned with the transmission axes of the polarisers. Rotate the waveplate by $\pi/4$. It must now produce circularly polarised light, which can be verified by rotating the analyser; very little variation in intensity should be observed. Any such variation is the result of the ellipticity of the light, and the ratio of maximum and minimum observed intensities is a measure of this.

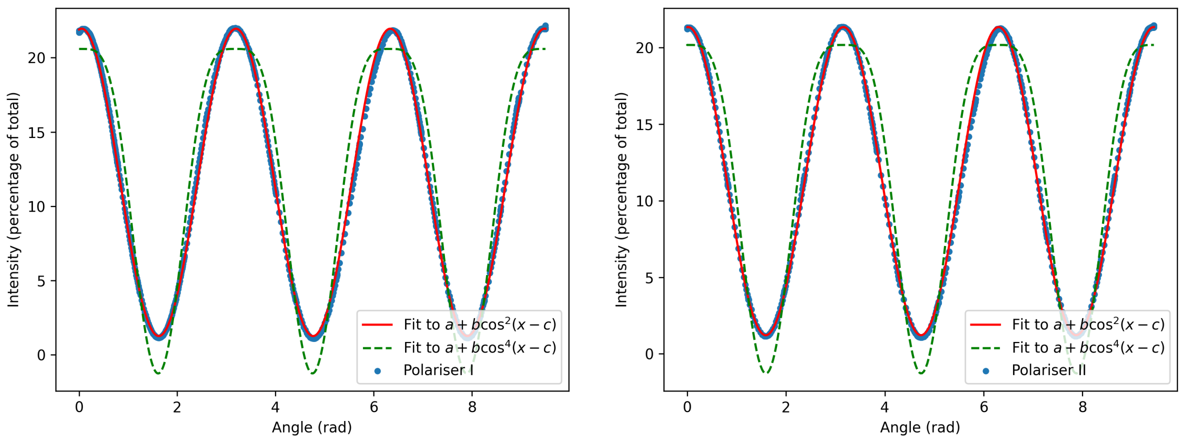
3 Experimental data and analysis

3.1 Processing and plotting

All data has been gathered into an Excel spreadsheet, read using `pandas` and processed using `numpy`. The code used has been listed below.

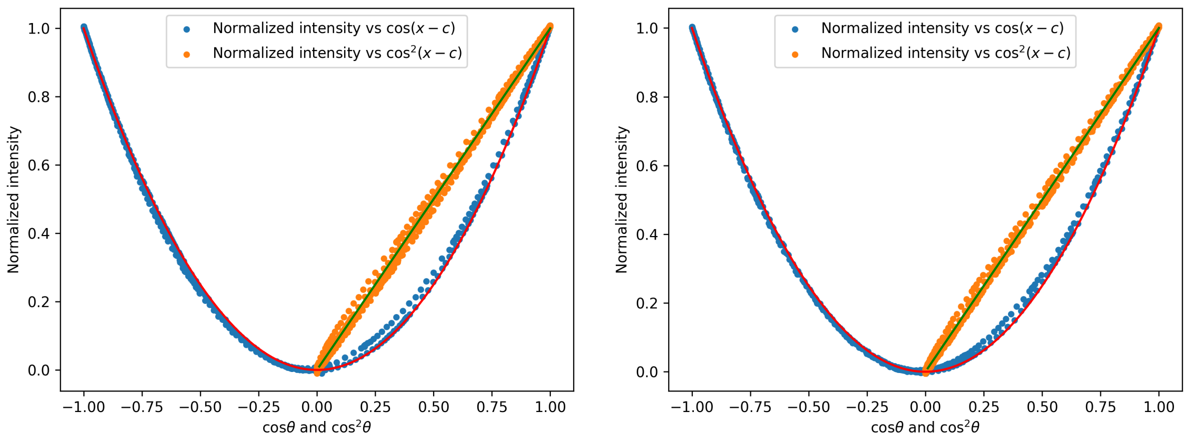
```
#!/usr/bin/env python3

import pandas as pd
import matplotlib.pyplot as plt
import numpy as np
from scipy import optimize

cos2 = lambda x, a, b, c: a + b * np.cos(x - c)**2
cos4 = lambda x, a, b, c: a + b * np.cos(x - c)**4
coscos = lambda x, a, b, c, d, e, f: (a + b * np.cos(x - c)) * (d + e * np.cos(
    x - f))

sheets = pd.read_excel('data.xlsx', sheet_name=None)
polarisers = ['Polariser␣I', 'Polariser␣II']
waveplates = ['Waveplate␣I', 'Waveplate␣II']

for name in polarisers:
    sheet = sheets[name]
    angle, intensity = sheet['Angle'], sheet['Intensity']
    coeff2, cov2 = optimize.curve_fit(cos2, angle, intensity)
    coeff4, cov4 = optimize.curve_fit(cos4, angle, intensity)
    plt.scatter(angle, intensity, label=name, s=12)
    plt.plot(angle, cos2(angle, *coeff2), '-r', label="Fit␣to␣$a␣+␣b\cos^2(x␣-␣
        c)$")
```

```
    plt.plot(angle, cos4(angle, *coeff4), '--g', label="Fit to $a + b\cos^4(x - c)$")
    print(coeff2, np.sqrt(np.diag(cov2)))
    plt.xlabel("Angle (rad)")
    plt.ylabel("Intensity (percentage of total)")
    plt.legend(loc="lower right")
    plt.show()

    a, b, c = coeff2
    x = np.linspace(-1, 1, 100)
    normalized = (intensity - a) / b
    plt.scatter(np.cos(angle - c), normalized, label='Normalized intensity vs $\cos(x - c)$', s=12)
    plt.plot(x, x**2, '-r')
    plt.scatter(np.cos(angle - c) ** 2, normalized, label='Normalized intensity vs $\cos^2(x - c)$', s=12)
    plt.plot(x[x > 0], x[x > 0], '-g')
    plt.xlabel("$\cos\\theta$ and $\cos^2\\theta$")
    plt.ylabel("Normalized intensity")
    plt.legend()
    plt.show()


for name in waveplates:
    sheet = sheets[name]
    angle, intensity = sheet['Angle'], sheet['Intensity']
    coeff, cov = optimize.curve_fit(coscos, angle, intensity)
    plt.scatter(angle, intensity, label=name, s=12)
    plt.plot(angle, coscos(angle, *coeff), '-r', label="Fit to $(a + b\cos(x - c))(d + e\cos(x - f))$")
    print(coeff, np.sqrt(np.diag(cov)))
    plt.xlabel("Angle (rad)")
    plt.ylabel("Intensity (percentage of total)")
    plt.legend(loc="lower right")
    plt.show()
```



(a) Polariser I (b) Polariser II

Figure 1: Intensities from the polariser-analyser setup as a function of the rotation angle. Note that the fit against $a + b\cos^2(x - c)$ is almost perfect, while the $a + b\cos^4(x - c)$ curve does not fit at all.



(a) Polariser I (b) Polariser II

Figure 2: The intensities from the polariser-analyser setup has been normalised as $I_{normal} = (I - a)/b$. These have been plotted against $\cos(x - c)$ and $\cos^2(x - c)$. The curves $y = x^2$ and $y = x$ have been drawn in red and green respectively to emphasize the $\cos^2$ relationship.

The fit parameters and uncertainties from Fig. 1 are as follows.

	a	b	c
Set I	21.95 ± 0.01	-20.69 ± 0.02	-7.80 ± 0.01
Set II	21.34 ± 0.01	-20.13 ± 0.01	-7.83 ± 0.01

Because of the manner in which the fit has been made ($b < 0$), we have $I_{max} = a$ and $I_{min} = a + b$. Thus,

$$\mathcal{P} = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{a - (a+b)}{a + (a+b)} = \frac{-b}{2a+b}, \qquad \mathcal{P}' = 1 - \frac{I_{min}}{I_{max}} = 1 - \frac{a+b}{a} = \frac{-b}{a}.$$

We calculate these for the two sets of data as follows.

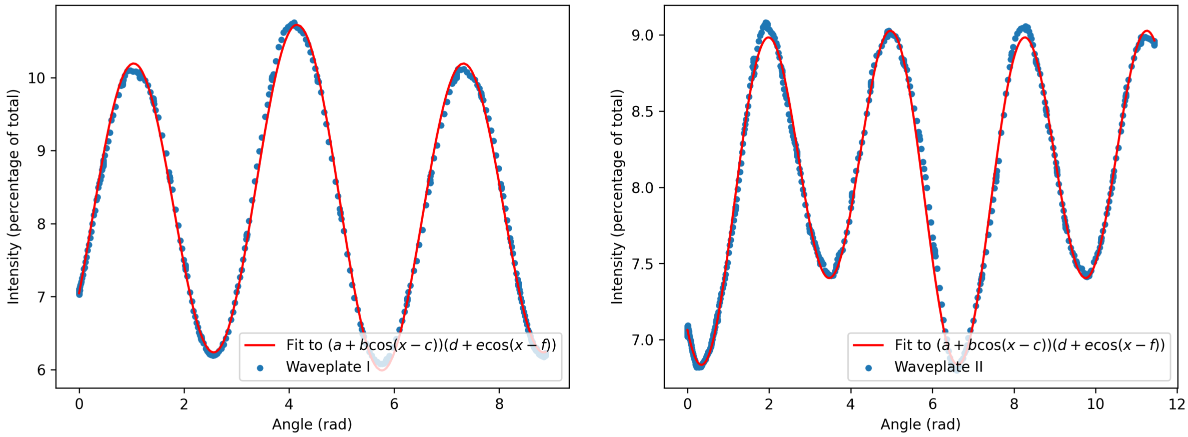
$$\mathcal{P}_I = 0.891, \qquad \mathcal{P}'_I = 0.943,$$

$$\mathcal{P}_{II} = 0.893, \qquad \mathcal{P}'_{II} = 0.943.$$

We estimate the ellipticity of the light from the waveplate using data from Fig. 3.

	I_{min}	I_{max}	$\eta = I_{min}/I_{max}$
Set I	6.05 ± 0.1	10.75 ± 0.1	0.563
Set II	6.80 ± 0.1	9.05 ± 0.1	0.751

Thus, neither set exhibits circularly polarised light, since I_{min}/I_{max} is not close to unity. Indeed, the particular pattern of alternating magnitudes of the maxima and minima cannot be explained by supposing elliptically polarised light. The fitting function $I = (a + b\cos(x-c))(d + e\cos(x-f))$ has been conjectured without a theoretical basis.



(a) Waveplate I

(b) Waveplate II

Figure 3: Intensities from the waveplate setup as a function of the rotation angle. Note that they do not fit an elliptically polarised light distribution, even allowing for imperfect polarisers. Thus, the light from the waveplate is not circularly polarised. The ellipticity is estimated as $\eta = I_{min}/I_{max}$.

3.2 Error Analysis

We write

$$(\delta\mathcal{P})^2 = \left|\frac{\partial\mathcal{P}}{\partial a}\right|^2 (\delta a)^2 + \left|\frac{\partial\mathcal{P}}{\partial b}\right|^2 (\delta b)^2 = \left(\frac{2b}{(2a+b)^2}\right)^2 (\delta a)^2 + \left(\frac{2a}{(2a+b)^2}\right)^2 (\delta b)^2,$$
$$(\delta\mathcal{P}')^2 = \left|\frac{\partial\mathcal{P}'}{\partial a}\right|^2 (\delta a)^2 + \left|\frac{\partial\mathcal{P}'}{\partial b}\right|^2 (\delta b)^2 = \left(\frac{b}{a^2}\right)^2 (\delta a)^2 + \left(\frac{1}{a}\right)^2 (\delta b)^2,$$
$$\left(\frac{\delta\eta}{\eta}\right)^2 = \left(\frac{\delta I_{min}}{I_{min}}\right)^2 + \left(\frac{\delta I_{max}}{I_{max}}\right)^2.$$

Thus, we calculate

$$\delta\mathcal{P}_I = 0.002, \qquad \delta\mathcal{P}'_I = 0.001,$$
$$\delta\mathcal{P}_{II} = 0.001, \qquad \delta\mathcal{P}'_{II} = 0.001.$$
$$\delta\eta_I = 0.011, \qquad \delta\eta_{II} = 0.013.$$

3.3 Reported Values

We see that the degree of polarisation of light produced is $\mathcal{P} = 89.2 \pm 0.2\%$ (or $\mathcal{P}' = 94.3 \pm 0.1\%$, depending on which definition is preferred).

We also note that the light produced by the waveplates is not circularly polarised. The measure of ellipticity η is not close to unity, instead it is 0.563 ± 0.011 in the first set and 0.751 ± 0.013 in the second.

4 Discussion

We see that the Malus Law holds very well, and any deviations can be explained by the imperfection of the polariser. The degree of polarisation of these polarisers is quite good.

The behaviour of the polarised light from the waveplate is unexpected, and requires deeper analysis to justify the form $(a + b\cos(x - c))(d + e\cos(x - f))$. This is not consistent with elliptically polarised light, and is certainly not with a circular polarisation. We may attempt to consider the effects of an imperfect first polariser, or an imperfect waveplate to try and explain the form of the intensity curve.

One hypothesis for the four extrema in the waveplate curve is that the polarisation of the light resembles an off-centre or squashed ellipse. The magnitude of a vector from the centre to the edge of such a shape goes through a periodic cycle with two different maxima and two minima, much like our intensity distribution. Such a polarisation may have arisen from an imperfect waveplate, perhaps tilted, or some external electric field which biases the oscillating vector in some particular direction.

4.1 Sources of error

The imperfections in the polariser comprise a source of systematic error. The least count of the rotary motion sensor introduces random error, as does any external lighting.

5 Conclusion

In conclusion, we have verified the Malus Law and have analysed the effect of waveplates on lieanrly polarised light.