PH2103: Physics Laboratory III

Interferometry and Coherence

Michelson's Interferometer and the refractive index of glass.

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Abstract

In this experiment, we employ division of amplitude interferometry in the form of Michelson's interferometer in order to determine the refractive index of glass. We also briefly discuss the concept of coherence.

1 Theory

Interferometry is the technique of using the phenomenon of interference to extract information, such as lengths on a microscopic scale, or the speed of light through a given medium. There are two primary classes of interferometers – division of wavefront and division of amplitude. We have already seen the former in the case of Young's Double Slit experiment. It is so named since a wavefront is divided into two distinct point 'sources' of coherent light, which recombine on a screen forming bands of dark and light lines called fringes. Division of amplitude interferometry is so called since a single beam is split into two beams at the same point in space, sent along different paths and recombined on a screen, forming linear or circular fringes. This is seen in the Michelson interferometer. It essentially consists of a source of monochromatic coherent light, a beamsplitter, two mirrors, and a screen. A slab of glass is often used as a compensator, which equalizes the optical path differences between the split beams. We use it here to control the path difference between the split beams, which can be used to measure the refractive index of the glass.

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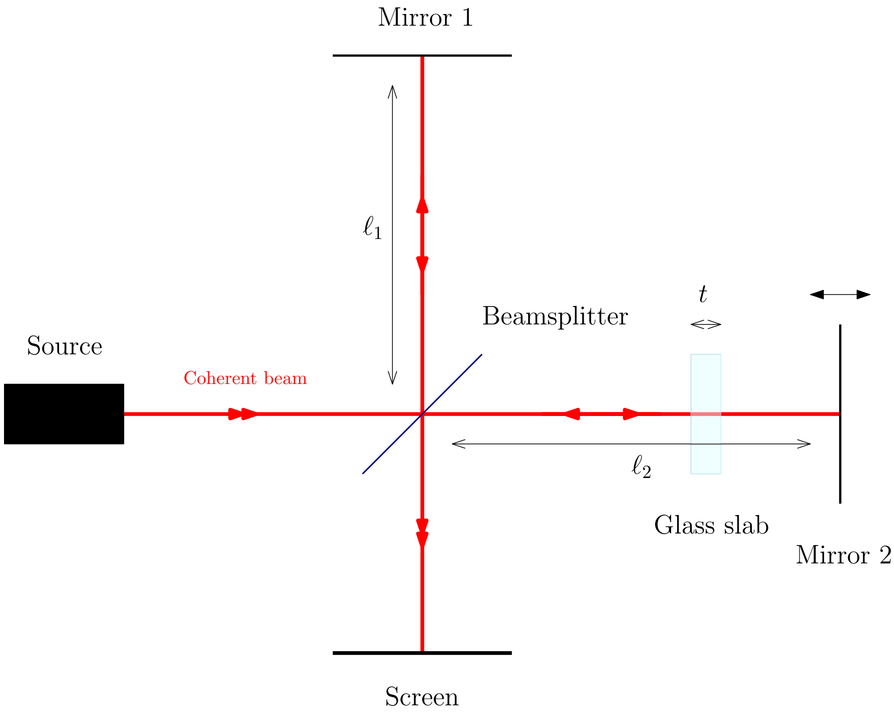

Figure 1: Schematic of a Michelson interferometer, with a glass slab inserted in one end. In our experiment, we change the path difference between the arms by rotating this slab, instead of moving the second mirror.

As seen in Fig. 1, the two beams cover different optical path lengths. The beam striking mirror 1 covers a distance $2\ell_1$ between the split and the recombination. The other beam covers a distance $2(\ell_2 - t)$ in air, and $2t$ in glass. Setting the refractive index of glass as n_g, this corresponds to an optical path of $2(\ell_2 - t) + 2n_g t$. Note the factors of 2, which arise because the beam travels to and back from the mirrors. The net optical path difference is thus $2\Delta\ell = 2(\ell_2 - \ell_1 + t(n_g - 1))$. The interference of these two beams with path difference $2\Delta\ell$ produces fringes on the screen. Note that the beams are actually spherical waves, so the path differences on the screen vary radially, producing circular fringes. Bright fringes are seen when the path difference is precisely an integer multiple of the wavelength λ of the light used. The center of the screen, marked by a crosswire, is a bright spot only if $2\Delta\ell = n\lambda$, for integral n.

Suppose that the path difference is changed by an amount $2d$. This can be done by moving one of the mirrors by d, or tilting the glass slab, effectively changing its thickness. This means that as the optical path difference at the center of the screen is changed from $2\Delta\ell \to 2\Delta\ell + 2d$, we move through several multiples of λ in between, i.e. bright fringes are formed and disappear from the center. The entire diffraction pattern moves inwards/outwards, and the number of bright fringes which pass the center is given by the number of multiples of λ in between the initial and final path differences, i.e.

$$d = \frac{m\lambda}{2}.$$

Refractive index calculation We know that a shift of m fringes corresponds to a path difference of $m\lambda/2$. Consider a glass slab of thickness t, with its face initially normal to the light beam. We have already seen that this contributes a path length of $(n_g - 1)t$ (one way).

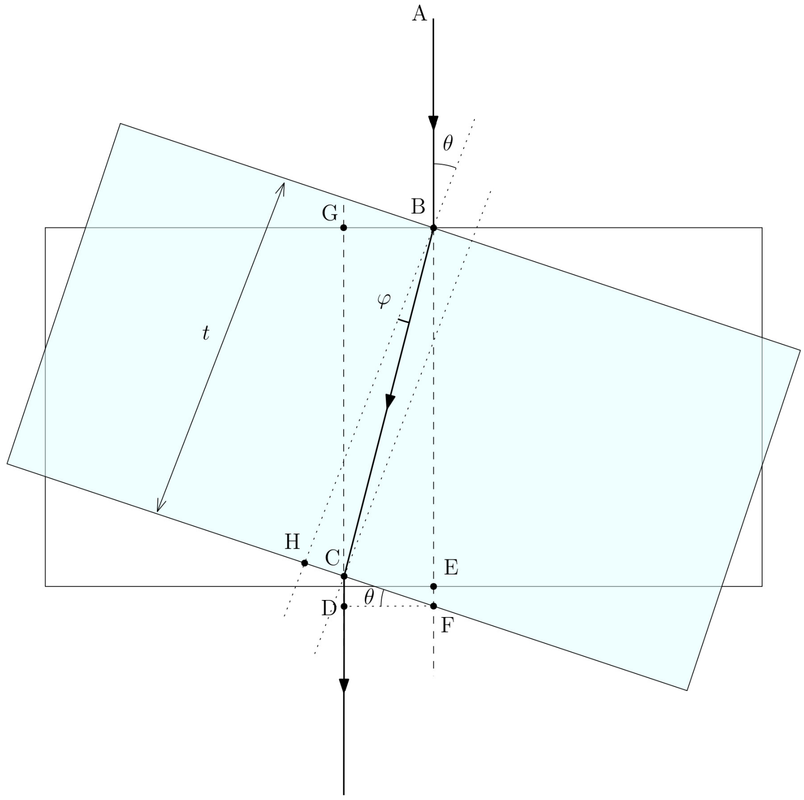

Figure 2: Schematic of the path followed by light through tthe rotated glass slab.

Upon rotating the slab by θ, light within the slab refracts and is deviated by an angle $\theta - \varphi$, where $\sin\theta = n_g \sin\varphi$ by Snell's Law. The old optical path $n_g BE + EF$ has been replaced with $n_g BC + CD$. Simple geometry gives $BE = t$, $EF = BF - BE = t/\cos\theta - t$, $BC = t/\cos\varphi$ and $CD = CF\sin\theta = (HF - HC)\sin\theta = t\tan\theta\sin\theta - t\tan\varphi\sin\theta$. Thus, the path difference is calculated as

$$\begin{aligned} n_g BC + CD - n_g BE - EF &= \frac{n_g t}{\cos\varphi} + \frac{t}{\cos\theta} + \frac{t\sin^2\theta}{\cos\theta} - \frac{t\sin\varphi\sin\theta}{\cos\varphi} - n_g t - \frac{t}{\cos\theta} + t \\ &= \frac{n_g t}{\cos\varphi}[1 - \cos\varphi - \sin\theta\sin\varphi/n_g] - \frac{t}{\cos\theta}[1 - \cos\theta - \sin^2\theta]. \end{aligned}$$

Simplifying using Snell's Law and equating with the inferred path difference from m fringe shifts, we obtain

$$\frac{m\lambda}{2t} = (1 - \cos\theta) - n_g(1 - \cos\varphi).$$

Rearranging,

$$1 - \cos\varphi = \frac{1}{n_g}\left[1 - \cos\theta - \frac{m\lambda}{2t}\right] = \frac{\alpha}{n_g}.$$

Now, this means that

$$\cos^2\varphi = \left[1 - \frac{\alpha}{n_g}\right]^2 = 1 - \frac{2\alpha}{n_g} + \frac{\alpha^2}{n_g^2}.$$

Using $\cos^2\varphi = 1 - \sin^2\varphi = 1 - \sin^2\theta/n_g^2$, we obtain $\sin^2\theta = 2\alpha n_g - \alpha^2$. Rearranging,

$$\begin{aligned} n_g &= \frac{\alpha^2 + \sin^2\theta}{2\alpha} \\ &= \frac{(1-\cos\theta)^2 - 2(1-\cos\theta)(m\lambda/2t) + (m\lambda/2t)^2 + \sin^2\theta}{2(1-\cos\theta - (m\lambda/2t))} \\ &= \frac{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta - 2(1-\cos\theta)(m\lambda/2t) + (m^2\lambda^2/4t^2)}{2(1-\cos\theta - (m\lambda/2t))}. \end{aligned}$$

Thus,

$$n_g = \frac{(1-\cos\theta)(1 - m\lambda/2t) + m^2\lambda^2/8t^2}{1 - \cos\theta - m\lambda/2t}.$$

For calculations, we can ignore the quadratic term, approximating

$$n_g = \frac{(1-\cos\theta)(1 - m\lambda/2t)}{1 - \cos\theta - m\lambda/2t}. \qquad (\star)$$

1.1 Coherence

The formation of interference fringes relies on the fact that the path/phase differences between the interfering waves is well-defined and known at all points on the screen. If two waves are in phase at the source, we must be confident that they remain in phase at the beamsplitter, are precisely $2\Delta\ell \cdot 2\pi/\lambda$ out of phase after recombination, and remain so at the screen. In effect, given the phase relationship of a wavetrain at a particular point of time and space, we must be able to say with confidence that this phase relationship is preserved over intervals of time and space. An ideal monochromatic, coherent light wave has an electric field of the form

$$U(t) = A\cos(kx - \omega t).$$

Note that this wave has exactly one frequency, ω.

In reality, this is not true – waves are not generated as infinite wavetrains, but in bursts, which may remain correlated over the duration and length of the burst, but not between bursts. This means that after a certain distance, i.e. when the wave has traveled for a certain time, the phase relationship in the beam becomes completely uncertain. This length is called the coherence length ℓ_c, and this time is called the coherence time τ_c, related as $\ell_c = c\tau_c$. Thus, if the path length of light in our setup approaches and begins to exceed ℓ_c, the fringes on the screen get washed out, eventually becoming unifrom. This is because the phase differences between waves at the screen essentially becomes random, so there are no particular regions where the intensities add up or cancel, hence no distinct maxima or minima.

This leads to the idea of spatial and temporal coherence. Consider a beam of light with planar wavefronts. It is expected that over lengths of order ℓ_c, these wavefronts will remain planar and maintain their shape. This is thus a measure of spatial coherence. It is also expected that the minima and maxima of the wavefront amplitudes will remain regular, i.e. equally spaced, over a time period τ_c. This is a measure of the temporal coherence of the wave.

To get a sense of scale for ℓ_c, consider two wavetrains originating at the same point in space, of wavelengths λ and $\lambda + \Delta\lambda$. Over short lengths, these appear to be identical, yet with distance, they fall out of phase. There will be a length ℓ where for the first time, the waves are exactly π out of phase, i.e. they interfere destructively here. Suppose the first wave completes n cycles here, so $\ell = n\lambda$. The second will only be halfway through the previous cycle, so $\ell = (n - 1/2)(\lambda + \Delta\lambda)$. Equating these, we see that

$$\ell_c \sim \ell \approx \frac{\lambda^2}{2\Delta\lambda}.$$

A quasi-monochromatic, partially coherent wave has a spread of frequencies ω. This means that its Fourier transform looks like a sharp, but finitely thick, peak centered at the mean frequency $\bar{\omega}$ and having a width of $\Delta\omega$, which is a measure of the spectral width, i.e. range of frequencies/wavelengths of the light, which in turn is inversely a measure of the temporal coherence. The larger the $\Delta\omega$, the more this peak lowers, broadens and flattens out. Such light remains partially coherent over lengths $\ell \gg 2\pi c/\bar{\omega}$, and $\ell \ll 2\pi c/\Delta\omega$. Note that $2\pi c/\bar{\omega}$ is essentially the average wavelength $\bar{\lambda}$ of the beam, and $2\pi c/\Delta\omega$ is a measure of the distance beyond which the waves fall out of phase.

This is formalized by introducing the degree of temporal coherence γ during interference calculations. Suppose two monochromatic waves $E_1(t)$ and $E_2(t)$ fall on a scren. The intensity distribution on the screen at a particular point where the time difference is τ is given by

$$I = I_1 + I_2 + 2\Re\langle E_1^*(t)E_2(t+\tau)\rangle.$$

Here, $I_i = \langle |E_i(t)|^2\rangle$. We define

$$\gamma(t) = \frac{\langle E_1^*(t)E_2(t+\tau)\rangle}{\sqrt{I_1 I_2}} = |\gamma(\tau)|e^{i\alpha(\tau)}e^{-\bar{\omega}\tau}.$$

Typically, $\alpha(\tau)$ changes much faster than $|\gamma(\tau)|$, so we approximate

$$I = I_1 + I_2 + \sqrt{I_1 I_2}|\gamma(\tau)|.$$

The degree of coherence is related to the fringe visibility $\mathcal{V}$ as

$$\mathcal{V} = \frac{I_{max} - I_{min}}{I_{max} - I_{min}} = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2}|\gamma(\tau)|.$$

When $I_1 = I_2$, we have $\mathcal{V} = |\gamma(\tau)|$. Also note that in such a case,

$$I_{max} = 2I[1 + |\gamma(\tau)|], \qquad I_{min} = 2I[1 - |\gamma(\tau)|].$$

The factor $\gamma(\tau)$ is essentially a measure of the correlation between the two waves at a given point in space, also called the mutual coherence. When the beams are perfectly coherent, $|\gamma(\tau)| = 1$ and when they are perfectly incoherent, $|\gamma(\tau)| = 0$. The latter happens when $\tau \gg \tau_c$, so $\mathcal{V} = 0$. Typically, $|\gamma(\tau)|$ will decrease with increasing τ, i.e. the fringe visbility decays with increasing path difference.

In our setup, we have used a laser with a supposedly large coherence length, well beyond the tens of meters. Thus, we do not have to make any special adjustments to ensure that the path difference is low, and our observed fringes remain quite distinct.

2 Experimental setup

A Michelson interferometer is set up, much like in Fig. 1, with the glass slab on a rotational stage. The fringes as magnified using a lens. Now, the stage is rotated and the number of fringes shifted is counted as a function of the rotation angle θ. Specifically, the angle θ is noted for fixed number of fringe shifts such as 20, 30, 40 and 50. This data is used to calculated n_g using the working formula ($\star$) as derived earlier. It is given that the wavelength of the laser $\lambda = 650$ nm, and the thickness of the glass $t = 1$ mm.

3 Experimental data and analysis

3.1 Data processing

All data has been gathered into an Excel spreadsheet, read using `pandas` and processed using `numpy`. The code used has been listed below.

```python
#!/usr/bin/env python2
# -*- coding: utf-8 -*-

import pandas as pd
import numpy as np

# Set constants
wavelength = 650e-9
thickness = 1e-3

# Calculate the refractive index
def n_g(m, theta):
    d1 = 1 - np.cos(theta * np.pi / 180.0)
    d2 = m * wavelength / (2.0 * thickness)
    return d1 * (1 - d2) / (d1 - d2)

# Calculate the propagated error
def err(m, theta):
    xi = np.cos(theta * np.pi / 180.0)
    zeta = m * wavelength / (2.0 * thickness)
    dtheta = 1 * np.pi / 180.0
    dlam = 3e-9
    dt = 0.1e-3

    dxi = np.abs(np.sin(theta * np.pi / 180.0) * dtheta)
    dzeta = zeta * np.sqrt((dlam / wavelength)**2 + (dt / thickness)**2)
    A = zeta * (1 - zeta) / (1 - xi - zeta)**2 * dxi
    B = xi * (1 - xi) / (1 - xi - zeta)**2 * dzeta
    # return np.sqrt(A**2 + B**2)
    return dxi

sheets = pd.read_excel('data.xlsx', sheet_name=None)
for sheet, data in sheets.items():
    fringe, angle, refractive = list(data.columns)

    # Set calculated indices and errors
    data[refractive] = n_g(data[fringe], data[angle])
    errors = err(data[fringe], data[angle])
    data.insert(3, 'Error', errors)

    # Output means and standard deviations
    print sheet, np.mean(data[refractive]), np.std(data[refractive])
    for i in [20, 30, 40, 50]:
        print i, np.std(data[data[fringe] == i][angle])
    print np.sqrt(sum(errors**2)) / len(errors)
    print

# Display data in LaTeX tabular format
for sheet, data in sheets.items():
    print sheet
    print data.to_latex(index=False)
```

3.2 Error Analysis

Set $\xi = \cos\theta$ and $\zeta = m\lambda/2t$. Thus, using standard error propagation formulae, we have

$$(\delta n_g)^2 = \left|\frac{\partial n_g}{\partial \xi}\right|^2 (\delta\xi)^2 + \left|\frac{\partial n_g}{\partial \zeta}\right|^2 (\delta\zeta)^2 = \left(\frac{\zeta(1-\zeta)}{(1-\xi-\zeta)^2}\right)^2 (\delta\xi)^2 + \left(\frac{\xi(1-\xi)}{(1-\xi-\zeta)^2}\right)^2 (\delta\zeta)^2.$$

Now,

$$\delta\xi = \left|\frac{\partial \xi}{\partial \theta}\right| \delta\theta = |\sin\theta|\,\delta\theta, \qquad \text{and} \qquad \left(\frac{\delta\zeta}{\zeta}\right)^2 = \left(\frac{\delta\lambda}{\lambda}\right)^2 + \left(\frac{\delta t}{t}\right)^2.$$

The errors for N such calculated values are averaged as $\delta n_g = \sqrt{\sum \delta n_{g,i}}$ for each set. This is done numerically with the aforementioned code.

We have chosen $\delta\lambda = 3$ nm, $\delta\theta = 1^\circ = \pi/180$ rad, and $\delta t = 0.1$ mm. Note that the standard deviations of θ for each m are well within 1° each. For each set, we also take the maximum of this propagated error and the standard deviation of the calculated values.

3.3 Tabulated data

Table 1: Data from Set I.

Fringes shifted m	Angle rotated θ°	Refractive Index n_g	Propagated error
20	8.7	2.283473	0.741606
20	9.4	1.925645	0.423516
20	8.7	2.283473	0.741606
20	8.6	2.354855	0.815078
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20	8.7	2.283473	0.741606
20	8.6	2.354855	0.815078
30	9.6	3.259884	1.722282
30	11.0	2.109946	0.493008
30	11.7	1.865807	0.325026
30	11.9	1.812576	0.292756
30	11.9	1.812576	0.292756
30	11.6	1.894715	0.343205
30	12.4	1.701337	0.230382
40	14.1	1.736125	0.226681
40	12.8	2.069726	0.418611
40	13.6	1.840182	0.280709
40	12.7	2.106101	0.442821
40	13.1	1.972179	0.356868
40	12.7	2.106101	0.442821
40	13.0	2.002920	0.375825
50	15.0	1.880618	0.282957
50	14.8	1.928176	0.308297
50	15.1	1.858380	0.271455
50	15.8	1.726177	0.207653
50	15.4	1.797093	0.240903
50	14.8	1.928176	0.308297
50	16.0	1.694606	0.193577

Table 2: Data from Set II.

Fringes shifted m	Angle rotated $\theta°$	Refractive Index n_g	Propagated error
20	11.5	1.469199	0.140328
20	11.3	1.494670	0.152458
20	11.4	1.481657	0.146206
20	11.1	1.522506	0.166217
20	11.5	1.469199	0.140328
30	13.7	1.506533	0.137747
30	13.5	1.530233	0.147881
30	13.7	1.506533	0.137747
30	13.6	1.518160	0.142679
30	13.5	1.530233	0.147881
40	15.3	1.558726	0.146858
40	15.5	1.536048	0.137777
40	15.8	1.504755	0.125666
40	15.5	1.536048	0.137777
40	15.4	1.547193	0.142208
50	17.4	1.525469	0.125831
50	17.2	1.545219	0.133074
50	17.6	1.506861	0.119170
50	17.3	1.535195	0.129375
50	17.8	1.489304	0.113029

Table 3: Data from Set III.

Fringes shifted m	Angle rotated $\theta°$	Refractive Index n_g	Propagated error
20	8.8	2.218503	0.677627
20	8.6	2.354855	0.815078
20	8.9	2.159131	0.621570
30	10.4	2.435918	0.768221
30	10.8	2.202763	0.565452
30	10.5	2.370494	0.708326
40	12.7	2.106101	0.442821
40	12.9	2.035387	0.396347
40	12.6	2.144694	0.469212
50	14.8	1.928176	0.308297
50	14.2	2.101284	0.409060
50	15.1	1.858380	0.271455

3.4 Reported Values

We report the following refractive indices, standard errors and percentage errors for each of the three sets.

Set I	2.03 ± 0.31	15%
Set II	1.52 ± 0.03	2%
Set III	2.16 ± 0.17	8%

It is clear that the refractive index from set I and II agree with each other, but disagree with the value from set II. Despite having the largest number of measurements, set I has a very high standard error of 0.3. This is likely because of the datapoint marked in red in Table 1.

4 Discussion

4.1 Sources of error

The greatest contributer of measurement error is the angle measurement, which seems to have a fairly large range for a given fringe shift. The errors due to the wavelength and thickness of the slab are practically negligeable.

Systematic error may arise from an incorrectly calibrated angle measurement, i.e. the laser may not be perfectly normal to the slab face at $\theta = 0$. The glass slab might also not be perfectly cut or may be inhomogeneous, although these errors are probably too small to have any effect.

4.2 Measurement of coherence length

In order to measure the coherence length of the laser beam, we dispense with the glass slab and seek the path difference $2\Delta\ell$ at which the fringe visibility $\mathcal{V}$ becomes zero, i.e. the fringes lose all contrast. We must also choose the path difference such that if it were any shorter, the fringes appear again. At such a path difference, i.e. the maximum path difference at which fringes are still visible, we may conclude that $\ell_c \approx 2\Delta\ell$. Now, we know that ℓ_c for a laser is typically very high, so merely extending one arm even across a room while leaving the other in place may not produce a high enough $\Delta\ell$. Introducing a thick glass slab does increase the path length, but perhaps by an insufficient amount, only by $2(n_g - 1)t$. Thus, we propose taking the beam from one of the arms and reflecting it back and forth several times across two mirrors facing each other before returning it to the beamsplitter. This is difficult, as it requires keeping count of the number of reflections. Suppose the light beam is allowed to bounce between two parallel mirror planes of length L separated by a distance d, in a way that it enters this arrangement at an angle β from the mirror planes. Thus, when light bouncing off the mirrors has travelled the entire length L, it has covered a distance $L/\cos\beta$. By increasing β closer and closer to $\pi/2$, we can thus increase the path length arbitrarily. Once the beam reaches the other end, we can send it back along the same path with the original 'Mirror 1'.

5 Conclusion

In conclusion, we have observed the phenomenon of interference via division of amplitude, and used this to calculate the refractive index of glass. We have also discussed the phenomenon od coherence in this context.

References

P.K. Chakrabarti. *Geometrical and Physical Optics.* New Central Book Agency, 1997.