PH 2102 : Mechanics II

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Exercise 1 An atomic clock is placed in a plane that moves at the rate 400 m/s with respect to the earth (treated as an inertial frame). The clock measures a time interval of 3600 s. By how much is the time interval elongated to an observer on the earth?

Solution As suggested, we use the approximation

$$\gamma = (1 - \beta^2)^{-1/2} \approx 1 + \frac{1}{2}\beta^2,$$

where $\beta = v/c$, because we are given that $v \ll c$. Now, we know that the time interval Δt in the inertial frame of the earth and the time interval $\Delta t'$ in the frame moving with velocity v are related by the time dilation formula as

$$\Delta t = \gamma \Delta t' \approx \Delta t' + \frac{1}{2}\beta^2 \Delta t'.$$

Thus, the difference between these intervals is given by

$$\frac{1}{2}\beta^2 \Delta t' = \frac{400^2}{2 \times 9 \times 10^{16}} \times 3600 = 3.2 \times 10^{-9} \text{ s}.$$

The elongation is thus 3.2 nanoseconds.

Exercise 2 The motion of a medium influences the speed of light through it. Consider a light beam passing through water flowing horizontally with velocity v. The beam is directed along the flowing water. (a) If the speed in water is u in the laboratory frame, express u in terms of v and n (the refractive index of water). (b) Show that for $v \ll c$, this expression can be reduced to the form

$$u \approx \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right).$$

Solution Let the frame of the laboratory be S. The frame of the water S' moves with velocity v with respect to S. In this frame S', we know that the speed of light in the water is given by $u' = c/n$, where n is the refractive index of water. Transforming to frame S while noting that all quantities are directed along the same axis, we use the velocity addition formula to see that

$$u = \frac{u' + v}{1 + u'v/c^2} = \frac{c/n + v}{1 + v/nc}. \tag{a}$$

When $v \ll c$, we can approximate $(1 + v/nc)^{-1} = 1 - v/nc + v^2/n^2c^2 - \cdots \approx 1 - v/nc$. Thus, we obtain the first order approximation

$$u \approx \left(\frac{c}{n} + v\right)\left(1 - \frac{v}{nc}\right) = \frac{c}{n} + v - \frac{v}{n^2} - \cancel{\frac{v^2}{nc}} \approx \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right). \tag{b}$$

Exercise 3 Consider the relativistic form of Newton's Second Law. Assume that force is given by

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}.$$

in an inertial frame, where $\boldsymbol{p}$ is the momentum. Show that for $\boldsymbol{F}$ parallel to $\boldsymbol{v}$, which is the velocity of the object upon which the force acts,

$$F = \frac{m_0}{(1 - v^2/c^2)^{3/2}} \frac{dv}{dt}.$$

Solution We know that the momentum of a moving body is given by $\boldsymbol{p} = m\boldsymbol{v} = \gamma m_0 \boldsymbol{v}$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$. We calculate the quantity

$$\frac{d\gamma}{dt} = \frac{d}{dt}\frac{1}{\sqrt{1-v^2/c^2}} = -\frac{1}{2}\cdot\frac{1}{(1-v^2/c^2)^{3/2}}\cdot\left(-\frac{2v}{c^2}\right)\frac{dv}{dt} = \gamma^3\frac{v}{c^2}\frac{dv}{dt}.$$

Thus, we have

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = \frac{d(\gamma m_0 \boldsymbol{v})}{dt} = \gamma m_0 \frac{d\boldsymbol{v}}{dt} + m_0 \boldsymbol{v}\frac{d\gamma}{dt} = \gamma m_0 \frac{d\boldsymbol{v}}{dt} + \gamma^3 m_0 \frac{v}{c^2}\frac{dv}{dt}\,\boldsymbol{v}.$$

In this equation, consider the components directed along $\boldsymbol{v}$ and perpendicular to $\boldsymbol{v}$. We are given that $\boldsymbol{F}$ is directed along $\boldsymbol{v}$, and so is the very last term. Thus, $d\boldsymbol{v}/dt$ must also be directed along $\boldsymbol{v}$, i.e. the acceleration has no centripetal component. This means that we can write $d\boldsymbol{v}/dt = (dv/dt)\hat{\boldsymbol{v}}$, $\boldsymbol{F} = F\hat{\boldsymbol{v}}$ and $\boldsymbol{v} = v\hat{\boldsymbol{v}}$. Taking magnitudes,

$$F = \gamma m_0 \frac{dv}{dt}\left(1 + \gamma^2 \frac{v^2}{c^2}\right) = \gamma m_0 \frac{dv}{dt}\left(1 + \frac{v^2/c^2}{1-v^2/c^2}\right) = \gamma m_0 \frac{dv}{dt}\cdot\frac{1}{1-v^2/c^2}.$$

Thus, we obtain the desired equation,

$$F = \frac{m_0}{(1-v^2/c^2)^{3/2}}\frac{dv}{dt}.$$

Exercise 4 A body of rest mass m_0 moving with speed v collides with an identical body at rest, sticks to it, and the combined lump moves along the same direction. (a) What is the momentum (3-momentum) of the resulting lump? (b) Express the rest mass of the resulting lump in terms of m_0 and $\gamma = 1/\sqrt{1-v^2/c^2}$.

Solution We simply calculate the four momenta $(E/c, \boldsymbol{p})$ of the two bodies and add them together, since the conservation of energy and 3-momenta guarantees that the four momentum of the system must be conserved. Supposing all motion is along the x axis, the moving body initially has energy $E = \gamma m_0 c^2$ and momentum $\gamma m_0 v$, where $\gamma = 1/\sqrt{1-v^2/c^2}$. The stationary body initially has energy $m_0 c^2$ and zero momentum, as it's at rest. Thus,

$$p_f^\mu = (\gamma m_0 c, \gamma m_0 v, 0, 0) + (m_0 c, 0, 0, 0) = ((\gamma+1)m_0 c, \gamma m_0 v, 0, 0)\,.$$

Thus the 3-momentum of the final lump is simply the last three coordinates, so

$$\boldsymbol{p} = \gamma m_0 v\,\hat{\mathbf{i}}. \tag{a}$$

We know that the rest mass is simply given by the norm of p^μ multiplied by a factor of c. Thus,

$$\left\|p_f^\mu\right\|^2 = (\gamma+1)^2 m_0^2 c^2 - \gamma^2 m_0^2 v^2 = m_0^2\left[\gamma^2 c^2 + 2\gamma c^2 + c^2 - \gamma^2 v^2\right].$$

Now, $c^2 - v^2 = c^2(1 - v^2/c^2) = c^2/\gamma^2$. Thus, our expression simplifies to

$$\left\|p_f^\mu\right\|^2 = m_0 c^2\left[2 + 2\gamma\right].$$

Taking a square root and dividing by c, we obtain the final rest mass

$$m_{0f} \;=\; m_0\sqrt{2+2\gamma} \;\geq\; 2m_0. \tag{b}$$