PH 2102 : Mechanics II

Satvik Saha, `19MS154`, Group E   October 26, 2020

Exercise 12   Use the continuity equation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0,$$

to show that the net charge coming into a fixed volume is equal to the change in charge content.

Solution   Let V be an arbitrary fixed volume, with boundary ∂V. The current flowing through an elementary area dS is given by $\boldsymbol{j} \cdot \hat{\boldsymbol{n}} \, dS$, where $\hat{\boldsymbol{n}}$ is the unit normal to dS. Integrating this over the entire boundary gives us the net current flux,

$$\Phi \; = \; \int_{\partial V} \boldsymbol{j} \cdot \hat{\boldsymbol{n}} \, dS.$$

If $\hat{\boldsymbol{n}}$ is chosen to point outwards, this integral represents the rate at which charge exits the volume V across its boundary. The negation of this likewise represents the rate of entry of charge into V. Since ∂V is a closed boundary, we can use Gauss's Divergence Theorem to obtain

$$\Phi \; = \; \int_{V} \boldsymbol{\nabla} \cdot \boldsymbol{j} \, dV.$$

Applying the continuity equation, we have

$$\Phi \; = \; -\int_{V} \frac{\partial \rho}{\partial t} \, dV \; = \; -\frac{d}{dt} \int_{V} \rho \, dV \; = \; -\frac{d}{dt} Q_V.$$

We have interchanged the order of the integral and the derivative, and have used the fact that the integral of the charge density over a fixed volume is equal to the charge Q_V enclosed by that volume. Thus, we arrive at the integral form of the continuity equation,

$$\frac{d}{dt} Q_V \; + \; \int_{\partial V} \boldsymbol{j} \cdot \hat{\boldsymbol{n}} \, dS \; = \; 0.$$

Thus, the rate at which charge flows into a fixed volume is precisely balanced by the rate of increase in charge content.

Exercise 13   In the below illustration of Compton scattering, show that

$$\lambda_2 - \lambda_1 = \frac{h}{m_e}(1 - \cos\theta).$$
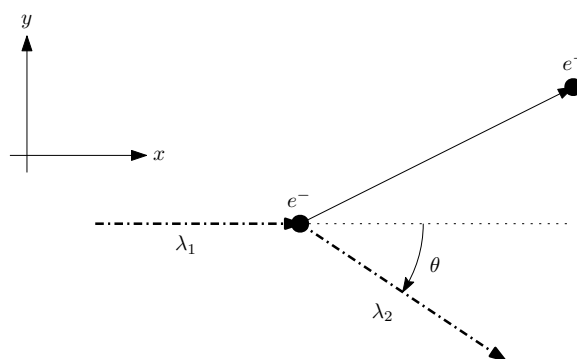
Solution The four momentum of a particle is given by $p^\mu = (E, \boldsymbol{p})$. The conservation of energy and 3-momentum can thus be expressed together as the conservation of four momentum. Note that

$$\|p^\mu\|^2 = p^\mu \cdot p^\mu = E^2 - \|\boldsymbol{p}\|^2 = m_0^2,$$

where m_0 is the rest mass of the particle. Thus, we calculate the momenta before and after the collision. The incident wave initially has energy h/λ_1, which is also equal to its momentum along the x-axis, since it has rest mass zero and $E^2 = m_0^2 + p^2$. After the collision, it has energy h/λ_2 which is also equal to its net momentum, directed at an angle θ to one side. Thus, its momenta along the x and y axes are $h \cos\theta/\lambda_2$ and $h \sin\theta/\lambda_2$. The electron initially has energy m_e (equal to its rest mass) and zero momentum as it was at rest. For the moment, let its four momentum after the collision be p_e^μ. We thus write

$$\underbrace{\left(\frac{h}{\lambda_1}, \frac{h}{\lambda_1}, 0, 0\right)}_{p_1^\mu} + \underbrace{(m_0, 0, 0, 0)}_{p_0^\mu} = \underbrace{\left(\frac{h}{\lambda_2}, \frac{h}{\lambda_2}\cos\theta, \frac{h}{\lambda_2}\sin\theta, 0\right)}_{p_2^\mu} + p_e^\mu.$$

Isolating p_e^μ to one side and taking square norms, we have

$$\|p_1^\mu - p_2^\mu + p_0^\mu\|^2 = \|p_e^\mu\|^2,$$
$$\|p_1^\mu\|^2 + \|p_2^\mu\|^2 + \|p_0^\mu\|^2 + 2p_1^\mu \cdot p_0^\mu - 2p_1^\mu \cdot p_2^\mu - 2p_2^\mu \cdot p_0^\mu = \|p_e^\mu\|^2.$$

Recall that $\|p_1^\mu\|^2 = \|p_2^\mu\|^2 = 0$, since the electromagnetic wave has zero rest mass. Furthermore, $\|p_0^\mu\|^2 = \|p_e^\mu\|^2 = m_e^2$, so these cancel on both sides. We are thus left with

$$(p_1^\mu - p_2^\mu) \cdot p_0^\mu = p_1^\mu \cdot p_2^\mu,$$
$$\left(\frac{h}{\lambda_1} - \frac{h}{\lambda_2}\right) m_e = \frac{h^2}{\lambda_1 \lambda_2} - \frac{h^2}{\lambda_1 \lambda_2}\cos\theta.$$
$$\frac{\lambda_2 - \lambda_1}{\lambda_1 \lambda_2} h m_e = \frac{h^2}{\lambda_1 \lambda_2}(1 - \cos\theta).$$

Cancelling like terms, we obtain the desired expression,

$$\lambda_2 - \lambda_1 = \frac{h}{m_e}(1 - \cos\theta).$$

Exercise 14 A body of rest mass m_0 moving at speed v collides with an identical body at rest. Thereafter, they stick together and move on. (a) Obtain the four-vector momentum of the final moving lump. (b) What is the rest mass of the resultant lump?

Solution We simply calculate the four momenta of the the two bodies and add them together, since the conservation of energy and 3-momenta guarantees that the four momentum of the system must be conserved. Supposing all motion is along the x axis, the moving body initially has energy $E = \gamma m_0 c^2$ and momentum $\gamma m_0 v$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$. The stationary body initially has energy $m_0 c^2$ and zero momentum, as it's at rest. Thus,

$$p_f^\mu = (\gamma m_0 c, \gamma m_0 v, 0, 0) + (m_0 c, 0, 0, 0) = ((\gamma + 1)m_0 c, \gamma m_0 v, 0, 0). \tag{a}$$

We have already shown that the rest mass is simply given by the norm of p^μ, up to a factor of c. Thus,

$$\left\|p_f^\mu\right\|^2 = (\gamma + 1)^2 m_0^2 c^2 - \gamma^2 m_0^2 v^2 = m_0^2 \left[\gamma^2 c^2 + 2\gamma c^2 + c^2 - \gamma^2 v^2\right].$$

Now, $c^2 - v^2 = c^2(1 - v^2/c^2) = c^2/\gamma^2$. Thus, our expression simplifies to

$$\left\|p_f^\mu\right\|^2 = m_0 c^2 \left[2 + 2\gamma\right].$$

Taking a square root and dividing by c, we obtain the final rest mass

$$m_{0f} = m_0 \sqrt{2 + 2\gamma} \geq 2m_0. \tag{b}$$

Exercise 15 The earth and the sun are 8.3 light minutes apart. Ignore their relative motion and suppose that they are in the same intertial frame. Event A occurs at $t = 0$ on the earth, and event B occurs at $t = 2$ minutes on the sun. Find the time difference between the events according to an observer moving at speed $0.8\,c$ from the earth to the sun.

Solution We set up our coordinate frames such that the x axis extends from the earth to the sun. In the first frame, we observe a time separation $\Delta t = 2$ minutes and a spatial separation of $\Delta x = 8.3$ light minutes. Our moving frame has velocity $0.8\,c$ along the positive x axis; its γ factor is given by $1/\sqrt{1 - 0.8^2} = 5/3$. Using the Lorentz transformation $t' = \gamma(t - vx/c^2)$, we obtain the temporal separation in the moving frame

$$\Delta t' = \gamma(\Delta t - v\Delta x/c) = \frac{5}{3}\,(2\text{ min} - (0.8)(8.3)\text{ min}) = -7.73\text{ minutes.}$$

Thus, in the moving frame, the event on the sun occurs 7.33 minutes *before* that on the earth. This reversal of order suggests that the two events are not causally linked, which is evident from the fact that the events in the static frame occur too soon after another (2 minutes) for a signal to travel between them (at least 8.3 minutes).

Exercise 16 An observer O who is on the x axis of a frame records a flash of red light at $x_R = 1210$ m, and after $4\,\mu$s, a flash of blue light at $x_B = 480$ m.

(a) What is the velocity (relative to O) of an observer O' moving along the x axis who records the events as occurring at the same spatial point?

(b) Which flash occurs first according to O' and what is the time interval between the two flashes as measured by O'?

Solution Suppose the observer moves with speed v, and let this moving frame be recorded with primed coordinates. We use the Lorentz transformation $x' = \gamma(x - vt)$ to conclude that when $\Delta x' = 0$, we demand

$$v = \frac{\Delta x}{\Delta t} = \frac{480 - 1210}{4\,\mu s} = -182.5 \times 10^6 \text{ m/s} = -0.61\,c. \tag{a}$$

Recall that since a signal can travel between the two events ($v < c$), the order of the events must be preserved between the frames, i.e. the red flash occurs first according to observer O'. In order to obtain the temporal separation, we use the invariance of the spacetime interval $c^2(\Delta t)^2 - (\Delta x)^2$, so

$$(\Delta t')^2 = (\Delta t)^2 - (\Delta x)^2/c^2 = 16 \times 10^{-12} - 5.93 \times 10^{-12} = 10.07 \times 10^{-12} \text{ s.}$$

Thus,

$$\Delta t' = 3.17\,\mu s. \tag{b}$$