PH 2102 : Mechanics II

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Exercise 9 Show that $u^\mu \cdot u^\mu = 1$, where $u^\mu = (\gamma,\, \gamma \boldsymbol{v}/c)$.

Solution From the definition of the dot product,

$$\begin{aligned} u^\mu \cdot u^\mu &= u^0u^0 - u^1u^1 - u^2u^2 - u^3u^3 \\ &= \gamma^2 - \gamma^2(v_x^2 + v_y^2 + v_z^2)/c^2 \\ &= \gamma^2(1 - v^2/c^2) \\ &= 1. \end{aligned}$$

The last line follows since $\gamma = 1/\sqrt{1 - v^2/c^2}$.

Exercise 10 Show that $u^\mu \cdot a^\mu = 0$, where $a^\mu = du^\mu/ds$.

Solution We know that $x^\mu = (ct,\, x,\, y,\, z)$. In the frame of proper time, the particle is at rest, so from the invariance of the spacetime interval ds,

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2d\tau^2.$$

This means that $ds = c\,d\tau$, where τ is the proper time. Also, $d\tau = dt\sqrt{1 - (dx^2 + dy^2 + dz^2)/dt^2} = dt\sqrt{1 - v^2/c^2}$, where v is the speed of the body. Thus, using the differential $dx^\mu = (c\,dt, dx, dy, dz)$, we have

$$u^\mu = \frac{d}{ds}x^\mu = \frac{d\tau}{ds}\frac{dt}{d\tau}\frac{d}{dt}x^\mu = \frac{\gamma}{c}\,(c,\, v_x,\, v_y,\, v_z) = \left(\gamma,\, \frac{\gamma}{c}\boldsymbol{v}\right).$$

To proceed further, we will need to calculate $d\gamma/dt$.

$$\frac{d}{dt}\frac{1}{\sqrt{1 - v^2/c^2}} = -\frac{1}{2}\cdot\frac{1}{(1 - v^2/c^2)^{3/2}}\cdot\left(-\frac{2\boldsymbol{v}}{c^2}\right)\cdot\frac{d}{dt}\boldsymbol{v} = \frac{\gamma^3}{c^2}\boldsymbol{v}\cdot\boldsymbol{a}.$$

This follows since $v^2 = \boldsymbol{v}\cdot\boldsymbol{v}$. The quantity $\boldsymbol{a}$ is the ordinary acceleration in 3 space. We take the derivative of u^μ in the same manner as before, obtaining

$$a^\mu = \frac{d}{ds}u^\mu = \frac{\gamma}{c}\frac{d}{dt}\left[\frac{\gamma}{c}(c,\, \boldsymbol{v})\right] = \frac{\gamma}{c^2}\frac{d\gamma}{dt}(c,\, \boldsymbol{v}) + \frac{\gamma^2}{c^2}\frac{d}{dt}(c,\, \boldsymbol{v}) = \frac{\gamma^4}{c^4}\boldsymbol{v}\cdot\boldsymbol{a}(c,\, \boldsymbol{v}) + \frac{\gamma^2}{c^2}(0,\, \boldsymbol{a}).$$

Simplifying, we obtain

$$a^\mu = \left(\frac{\gamma^4}{c^3}\boldsymbol{v}\cdot\boldsymbol{a},\; \frac{\gamma^4}{c^4}(\boldsymbol{v}\cdot\boldsymbol{a})\boldsymbol{v} + \frac{\gamma^2}{c^2}\boldsymbol{a}\right).$$

Thus, we see that

$$u^\mu \cdot a^\mu = \frac{\gamma^5}{c^3}\boldsymbol{v}\cdot\boldsymbol{a} - \frac{\gamma^5}{c^5}(\boldsymbol{v}\cdot\boldsymbol{a})(\boldsymbol{v}\cdot\boldsymbol{v}) - \frac{\gamma^3}{c^3}(\boldsymbol{v}\cdot\boldsymbol{a}) = \frac{\gamma^3}{c^3}(\boldsymbol{v}\cdot\boldsymbol{a})\left[\gamma^2 - \gamma^2\frac{v^2}{c^2} - 1\right] = 0.$$

The last equality follows because $\gamma^2(1 - v^2/c^2) = 1$.

Exercise 11 Show that

$$mc\frac{d}{ds}x^\mu \;=\; p^\mu.$$

Solution We know that $dx^\mu = (c\,dt,\, dx,\, dy,\, dz)$. We also know that if τ represents the proper time, then $ds = c\,d\tau$. Thus,

$$\frac{d}{ds}x^\mu = \frac{d\tau}{ds}\frac{d}{d\tau}x^\mu = \frac{1}{c}\left(c\frac{dt}{d\tau},\, \frac{dx}{d\tau},\, \frac{dy}{d\tau},\, \frac{dz}{d\tau}\right).$$

In addition, we know that $d\tau = dt\sqrt{1 - v^2/c^2} = dt/\gamma$, where $v^2 = (dx^2 + dy^2 + dz^2)/dt^2$ is the square of the speed of the particle m. Thus,

$$\frac{dx}{d\tau} = \frac{dt}{d\tau}\frac{dx}{dt} = \gamma v_x, \qquad \frac{dy}{d\tau} = \frac{dt}{d\tau}\frac{dy}{dt} = \gamma v_y, \qquad \frac{dz}{d\tau} = \frac{dt}{d\tau}\frac{dz}{dt} = \gamma v_z.$$

Thus, since $\boldsymbol{p} = m\gamma\boldsymbol{v}$, we have

$$mc\frac{d}{ds}x^\mu = (\gamma mc,\, \gamma m v_x,\, \gamma m v_y,\, \gamma m v_z) = \left(\frac{E}{c},\, \boldsymbol{p}\right) = p^\mu.$$

The second equality follows since $E = \gamma mc^2$.