PH 2102 : Mechanics II

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Exercise 1 Explicitly prove that the two ways of writing $\boldsymbol{A}\cdot\boldsymbol{B}$ and $\boldsymbol{A}\times\boldsymbol{B}$ are equivalent.

$$\begin{aligned}
\boldsymbol{A}\cdot\boldsymbol{B} &= A_xB_x + A_yB_y + A_zB_z, \\
\boldsymbol{A}\cdot\boldsymbol{B} &= \|\boldsymbol{A}\|\,\|\boldsymbol{B}\|\cos\theta. \\
\boldsymbol{A}\times\boldsymbol{B} &= (A_yB_z - A_zB_y)\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\hat{\mathbf{k}}, \\
\|\boldsymbol{A}\times\boldsymbol{B}\| &= \|\boldsymbol{A}\|\,\|\boldsymbol{B}\|\sin\theta.
\end{aligned}$$

Solution First, we show that

$$\boldsymbol{A}\cdot\boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z = \|\boldsymbol{A}\|\,\|\boldsymbol{B}\|\cos\theta$$

Start with $\boldsymbol{A}\cdot\boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z$. Then, $\|\boldsymbol{A}\|^2 = \boldsymbol{A}\cdot\boldsymbol{A} = A_x^2 + A_y^2 + A_z^2$ and $\|\boldsymbol{B}\|^2 = \boldsymbol{B}\cdot\boldsymbol{B} = B_x^2 + B_y^2 + B_z^2$.
Construct $\boldsymbol{C} = \boldsymbol{A} - \boldsymbol{B}$. Thus, the lengths $\|\boldsymbol{A}\|$, $\|\boldsymbol{B}\|$ and $\|\boldsymbol{C}\|$ form a triangle. Applying the cosine rule yields

$$\|\boldsymbol{C}\|^2 = \|\boldsymbol{A}\|^2 + \|\boldsymbol{B}\|^2 - 2\,\|\boldsymbol{A}\|\,\|\boldsymbol{B}\|\cos\theta.$$

On the other hand, by the laws of vector arithmetic, we have

$$\begin{aligned}
\|\boldsymbol{C}\|^2 &= \boldsymbol{C}\cdot\boldsymbol{C} \\
&= (\boldsymbol{A}-\boldsymbol{B})\cdot(\boldsymbol{A}-\boldsymbol{B}) \\
&= \boldsymbol{A}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot\boldsymbol{B} - \boldsymbol{B}\cdot\boldsymbol{A} + \boldsymbol{B}\cdot\boldsymbol{B} \\
\|\boldsymbol{C}\|^2 &= \|\boldsymbol{A}\|^2 + \|\boldsymbol{B}\|^2 - 2\boldsymbol{A}\cdot\boldsymbol{B}.
\end{aligned}$$

Comparing and equating the two expressions for $\|\boldsymbol{C}\|^2$, we obtain $\boldsymbol{A}\cdot\boldsymbol{B} = \|\boldsymbol{A}\|\,\|\boldsymbol{B}\|\cos\theta$.
Now, start with

$$\boldsymbol{A}\times\boldsymbol{B} = (A_yB_z - A_zB_y)\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\hat{\mathbf{k}}.$$

Taking norms and squaring,

$$\begin{aligned}
\|\boldsymbol{A}\times\boldsymbol{B}\|^2 &= \sum_{cyc}(A_yB_z - A_zB_y)^2 \\
&= \sum_{cyc}(A_y^2B_z^2 + A_z^2B_y^2 - 2A_yA_zB_yB_z) \\
&= A_x^2(B_y^2+B_z^2) + A_y^2(B_x^2+B_z^2) + A_z^2(B_x^2+B_y^2) - 2\sum_{cyc}A_yA_zB_yB_z \\
&= (A_x^2+A_y^2+A_z^2)(B_x^2+B_y^2+B_z^2) - (A_x^2B_x^2 + A_y^2B_y^2 + A_z^2B_z^2) - 2\sum_{cyc}A_yB_yA_zB_z \\
&= (A_x^2+A_y^2+A_z^2)(B_x^2+B_y^2+B_z^2) - (A_xB_x + A_yB_y + A_zB_z)^2.
\end{aligned}$$

Also, note that

$$\begin{aligned}
\|\boldsymbol{A}\|^2\,\|\boldsymbol{B}\|^2\sin^2\theta &= \|\boldsymbol{A}\|^2\,\|\boldsymbol{B}\|^2(1-\cos^2\theta) \\
&= \|\boldsymbol{A}\|^2\,\|\boldsymbol{B}\|^2 - (\boldsymbol{A}\cdot\boldsymbol{B})^2 \\
&= (A_x^2+A_y^2+A_z^2)(B_x^2+B_y^2+B_z^2) - (A_xB_x + A_yB_y + A_zB_z)^2.
\end{aligned}$$

Thus, we have $\|\boldsymbol{A}\times\boldsymbol{B}\| = \|\boldsymbol{A}\|\,\|\boldsymbol{B}\|\sin\theta$.

Exercise 2

(i) Show that $\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\phi) = \mathbf{0}$ when $\phi(\boldsymbol{r}) = 1/r$.

(ii) Show that $\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\phi) = \mathbf{0}$ in general for any $\phi(x, y, z)$.

(iii) Show that $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0$ in general for any $\boldsymbol{A}(x, y, z)$.

Solution

(i) Note that

$$\phi(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$

We calculate

$$\begin{aligned}
\boldsymbol{\nabla}\phi(x, y, z) &= \frac{\partial\phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{k}} \\
&= -\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\hat{\mathbf{i}} - \frac{y}{(x^2 + y^2 + z^2)^{3/2}}\hat{\mathbf{j}} - \frac{z}{(x^2 + y^2 + z^2)^{3/2}}\hat{\mathbf{k}}
\end{aligned}$$

Now,

$$\begin{aligned}
(\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi)_x &= \frac{\partial}{\partial y}\left[\frac{-z}{(x^2 + y^2 + z^2)^{3/2}}\right] - \frac{\partial}{\partial z}\left[\frac{-y}{(x^2 + y^2 + z^2)^{3/2}}\right] \\
&= \frac{3yz}{(x^2 + y^2 + z^2)^{5/2}} - \frac{3zy}{(x^2 + y^2 + z^2)^{5/2}} \\
&= 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi)_y &= \frac{\partial}{\partial z}\left[\frac{-x}{(x^2 + y^2 + z^2)^{3/2}}\right] - \frac{\partial}{\partial x}\left[\frac{-z}{(x^2 + y^2 + z^2)^{3/2}}\right] = 0. \\
(\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi)_z &= \frac{\partial}{\partial x}\left[\frac{-y}{(x^2 + y^2 + z^2)^{3/2}}\right] - \frac{\partial}{\partial y}\left[\frac{-x}{(x^2 + y^2 + z^2)^{3/2}}\right] = 0.
\end{aligned}$$

Thus, we have $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = \mathbf{0}$ in this instance.

(ii) We use the formula

$$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}.$$

Thus,

$$(\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi)_x = \frac{\partial}{\partial y}(\boldsymbol{\nabla}\phi)_z - \frac{\partial}{\partial z}(\boldsymbol{\nabla}\phi)_y = \frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y} = 0.$$

Similarly,

$$\begin{aligned}
(\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi)_y &= \frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z} = 0. \\
(\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi)_z &= \frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x} = 0.
\end{aligned}$$

Thus, we have $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = \mathbf{0}$ in general.

(iii) We have

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{k}}.$$

Thus,

$$\begin{aligned}
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) &= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\
&= \frac{\partial^2 A_x}{\partial y\partial z} - \frac{\partial^2 A_x}{\partial z\partial y} + \frac{\partial^2 A_y}{\partial z\partial x} - \frac{\partial^2 A_y}{\partial x\partial z} + \frac{\partial^2 A_z}{\partial x\partial y} - \frac{\partial^2 A_z}{\partial y\partial x} \\
&= 0.
\end{aligned}$$

Exercise 3 Prove the invariance of

$$\boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} \; = \; 0$$

for the same field configutation and charge movement as in the given example.

Solution Recall that in frame S, we have a charge q with velocity $\boldsymbol{v} = v\hat{\mathbf{\imath}}$ experiencing fields $\boldsymbol{E} = E\hat{\mathbf{\jmath}}$ and $\boldsymbol{B} = B\hat{\mathbf{k}}$. In the frame S' which moves with constant velocity $\boldsymbol{V} = V\hat{\mathbf{\imath}}$ with respect to S, the transformed quantities are $\boldsymbol{v}'$, $\boldsymbol{E}'$ and $\boldsymbol{B}'$. Special Relativity demands that the coordinates transform as follows.

$$\begin{aligned}
x' &= \gamma(x - Vt), \\
y' &= y, \\
z' &= z, \\
t' &= \gamma(t - Vx/c^2),
\end{aligned}$$

where $\gamma = 1/\sqrt{1 - V^2/c^2}$. Furthermore, we state without proof that the fields transform as

$$\begin{aligned}
E &= \gamma(E' + VB'), \\
B &= \gamma(B' + VE'/c^2).
\end{aligned}$$

In our given configuration, $E_x = E_z = 0$ and $B_z = B_y = 0$. Also, we are only concerned with the variation along $\hat{\mathbf{\imath}}$, so E, B are purely functions of x, t. Thus, all we must verify is that the following equation is invariant.

$$\frac{\partial E}{\partial x} + \frac{\partial B}{\partial t} \; = \; 0.$$

We calculate

$$\begin{aligned}
\frac{\partial x'}{\partial x} &= \gamma, & \frac{\partial x'}{\partial t} &= -\gamma V, \\
\frac{\partial t'}{\partial x} &= -\gamma \frac{V}{c^2}, & \frac{\partial t'}{\partial t} &= \gamma.
\end{aligned}$$

Thus, the differential operators transform as follows.

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} = \gamma \frac{\partial}{\partial x'} - \gamma \frac{V}{c^2}\frac{\partial}{\partial t'} \\
\frac{\partial}{\partial t} &= \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = -\gamma V \frac{\partial}{\partial x'} + \gamma \frac{\partial}{\partial t'}
\end{aligned}$$

We then substitute our formulae for E, B in terms of E', B' into the given equation, along with the transformed differential operators.

$$\begin{aligned}
\frac{\partial E}{\partial x} + \frac{\partial B}{\partial t} &= 0 \\
\left[\gamma \frac{\partial}{\partial x'} - \gamma \frac{V}{c^2}\frac{\partial}{\partial t'}\right](\gamma E' + \gamma V B') + \left[-\gamma V \frac{\partial}{\partial x'} + \gamma \frac{\partial}{\partial t'}\right]\left(\gamma B' + \gamma \frac{V}{c^2} E'\right) &= 0, \\
\left(\gamma^2 - \gamma^2 \frac{V^2}{c^2}\right)\frac{\partial E'}{\partial x'} + \left(-\gamma^2 \frac{V}{c^2} + \gamma^2 \frac{V}{c^2}\right)\frac{\partial E'}{\partial t'} + \left(\gamma^2 V - \gamma^2 V\right)\frac{\partial B'}{\partial x'} + \left(-\gamma^2 \frac{V^2}{c^2} + \gamma^2\right)\frac{\partial B'}{\partial t'} &= 0, \\
\left(\gamma^2 - \gamma^2 \frac{V^2}{c^2}\right)\frac{\partial E'}{\partial x'} + \left(-\gamma^2 \frac{V^2}{c^2} + \gamma^2\right)\frac{\partial B'}{\partial t'} &= 0.
\end{aligned}$$

Using the fact thet $\gamma^2(1 - V^2/c^2) = 1$ (assuming $V < c$), we obtain the transformed equation

$$\frac{\partial E'}{\partial x'} + \frac{\partial B'}{\partial t'} = 0,$$

as desired.