An Introduction to Statistical Depth Functions

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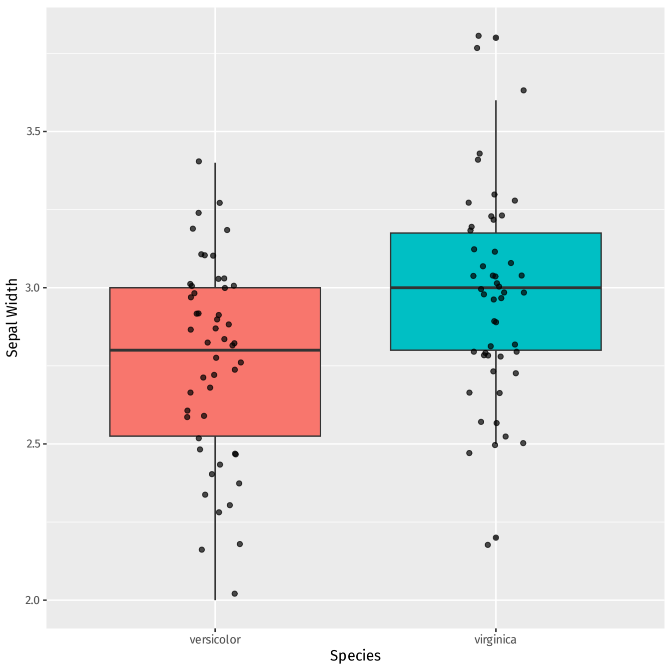
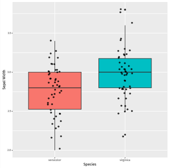
Outline

1. A two-sample testing problem
2. Depth Functions
3. The Depth-Depth plot
4. Depth based classification
5. Depth functions for Functional Data
6. Future work

A two-sample testing problem


Sepal Width
3.5
3.0
2.5
2.0
versicolor
virginica
Species



The figure illustrates the distribution of sepal widths from two species (‘versicolor’ and ‘virginica’), from the ‘Iris’ dataset in R.

Wilcoxon rank sum test

Given two random samples $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$, construct

$$W = \sum_j r(Y_j, \mathscr{D}_F \cup \mathscr{D}_G), \qquad r(Y, \mathscr{D}) = \sum_{Z \in \mathscr{D}} 1(Z \leq Y).$$

This is *distribution free* under the null hypothesis that both samples have the same underlying distribution.

2023-12-11

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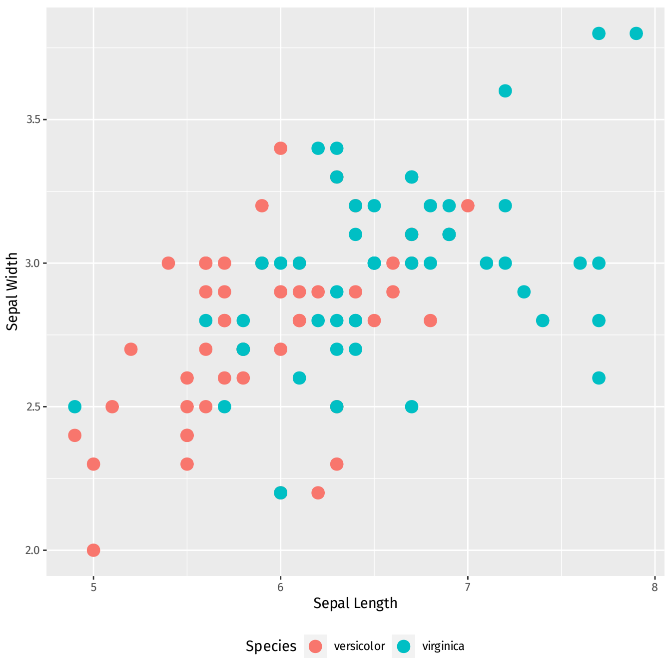
This is *distribution free* under the null hypothesis that both samples have the same underlying distribution.

The two-sided Wilcoxon rank sum test gives a *p*-value of 0.005, hence we reject the null hypothesis that the true location shift is zero.



Sepal Width
3.5
3.0
2.5
2.0
5
6
7
8
Sepal Length
Species
versicolor
virginica

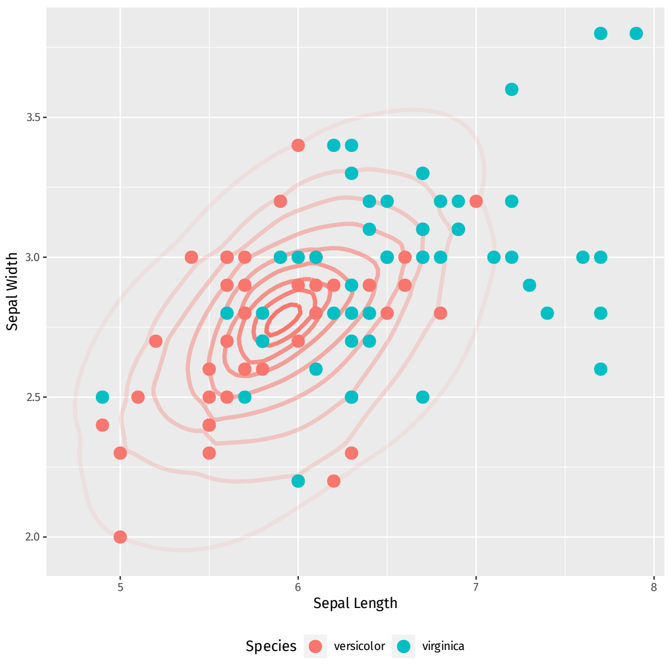
A generalization for multivariate data

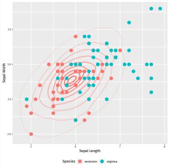
Given *multivariate* data, we wish to construct

$$W^* = \sum_j r(Y_j, \mathscr{D}_F \cup \mathscr{D}_G), \qquad r^*(Y, \mathscr{D}) = \sum_{Z \in \mathscr{D}} \mathbb{1}(Z \;??\; Y).$$

Furthermore, we want W^* to be able to detect differences in location and scale between F and G.

Liu, R.Y., & Singh, K. (1993) A Quality Index Based on Data Depth and Multivariate Rank Tests.


3.5
3.0
2.5
2.0
Sepal Width
5
6
7
8
Sepal Length
Species
versicolor
virginica



The red lines are spatial depth contours, drawn with reference to the ‘versicolor’ data.

How do we quantify this notion of centrality?

Depth Functions

Depth Functions

A *depth function* quantifies how central a point x is with respect to a distribution F.

Points which are *more central* are said to be *deeper*.

This framework allows many rank based nonparametric techniques to be translated to a broader class of data, e.g. multivariate and functional data.

2023-12-11

- Depth induces a *center-outwards* ordering of points.
- Contrast with the notion of rank which gives a *lowest-highest* ordering in the univariate setting.

Depth Functions

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Some applications of depth functions

1. Inference procedures
    - Hypothesis tests
    - Rank tests
    - Multivariate quantiles
    - Confidence regions
2. Exploratory data analysis
3. Classification and clustering
4. Outlier detection

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    - Hypothesis tests
    - Rank tests
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- Hypothesis tests – Two sample quality index
- Exploratory data analysis – D-D plots

Let $D\colon \mathbb{R}^p \times \mathscr{F} \to \mathbb{R}$ be bounded, non-negative, continuous, and satisfy the following properties.

1. **Affine invariance:** $D(A\mathbf{x} + b, F_{A\mathbf{x}+b}) = D(\mathbf{x}, F_X)$.
2. **Maximality at centre:** $D(\theta, F_X) = \sup_{\mathbf{x}\in\mathbb{R}^p} D(\mathbf{x}, F)$.
3. **Monotonicity along rays:** $D(\mathbf{x}, F) \leq D(\theta + \alpha(\mathbf{x} - \theta), F)$.
4. **Vanish at infinity:** $D(\mathbf{x}, F) \to 0$ as $\|\mathbf{x}\| \to \infty$.

Zuo, Y., & Serfling, R. (2000) General notions of statistical depth function

Depth contours

The *region of depth d* is defined by

$$\mathcal{R}(d, F) = \{\mathbf{x} \in \mathbb{R}^p \mid D(\mathbf{x}, F) \geq d\}.$$

The boundary $\partial\mathcal{R}(d, F)$ is called the *contour of depth d*.

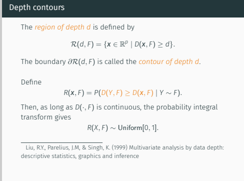
Define

$$R(\mathbf{x}, F) = P(D(Y, F) \geq D(\mathbf{x}, F) \mid Y \sim F).$$

Then, as long as $D(\cdot, F)$ is continuous, the probability integral transform gives

$$R(X, F) \sim \text{Uniform}[0, 1].$$

Liu, R.Y., Parelius, J.M, & Singh, K. (1999) Multivariate analysis by data depth: descriptive statistics, graphics and inference



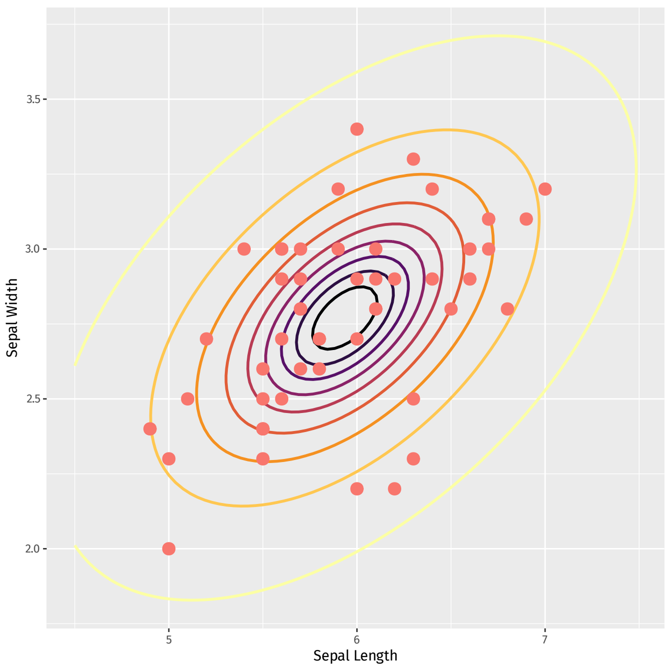
- Depth contours are analogous to univariate quantiles.
- Sample points ordered with respect to their corresponding $R(X_j, \hat{F}_n)$ are analogous to order statistics.
- The distribution-free nature of $R(X, F)$ is analogous to how $F(X) \sim \text{Uniform}[0, 1]$.

Mahalanobis depth

Produces elliptic contours, using the first two moments of the given distribution.

$$D_{Mh}(x, F) = \frac{1}{1 + (x - \mu)\Sigma^{-1}(x - \mu)}.$$

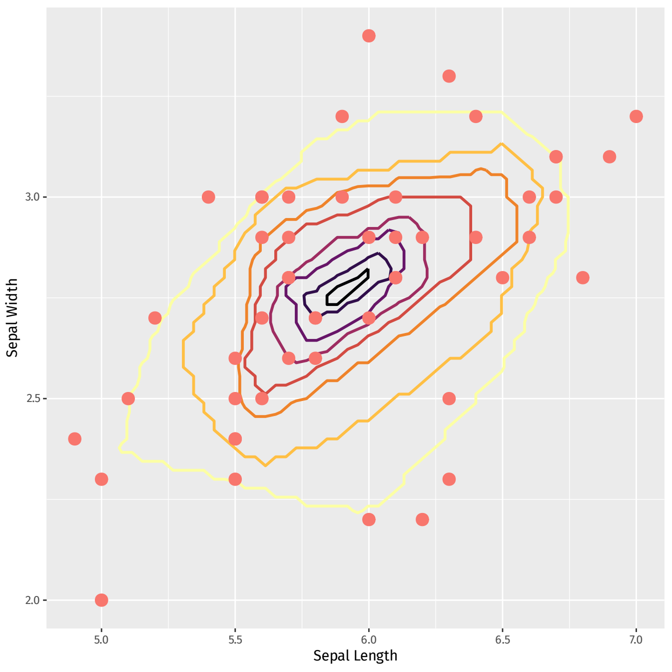
A robust version can be obtained by using MCD estimators.


3.5
3.0
2.5
2.0
Sepal Width
5
6
7
Sepal Length

Halfspace/Tukey depth

Given a point $\boldsymbol{x} \in \mathbb{R}^p$, examine all hyperplanes through $\boldsymbol{x}$, and find the halfspace with the least probability.

$$D_H(x, F) = \inf_{v \in \mathbb{R}^p \setminus \{0\}} P(\underbrace{v^\top X \leq v^\top x}_{\substack{X \text{ is in a halfspace} \\ \text{through } x}}).$$


Sepal Width
3.0
2.5
2.0
5.0
5.5
6.0
6.5
7.0
Sepal Length

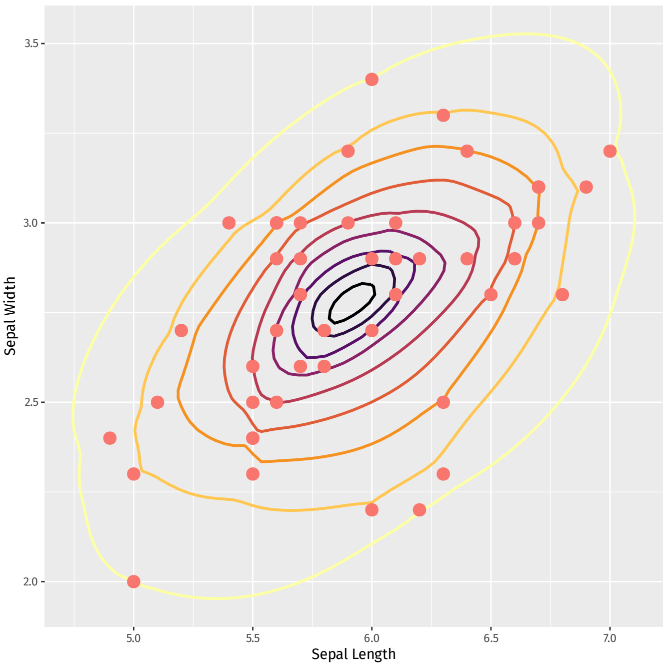
Spatial depth

Examine the average of unit vectors pointing out of ***x***.

$$D_{Sp}(x, F) = 1 - \left\| E\left[\underbrace{\frac{X - x}{\|X - x\|}}_{\substack{\text{unit vector} \\ \text{from } x \text{ to } X}} \right] \right\|.$$

Spatial depth is *not* always monotonic with respect to the deepest point.

Nagy., S. (2017) Monotonicity properties of spatial depth

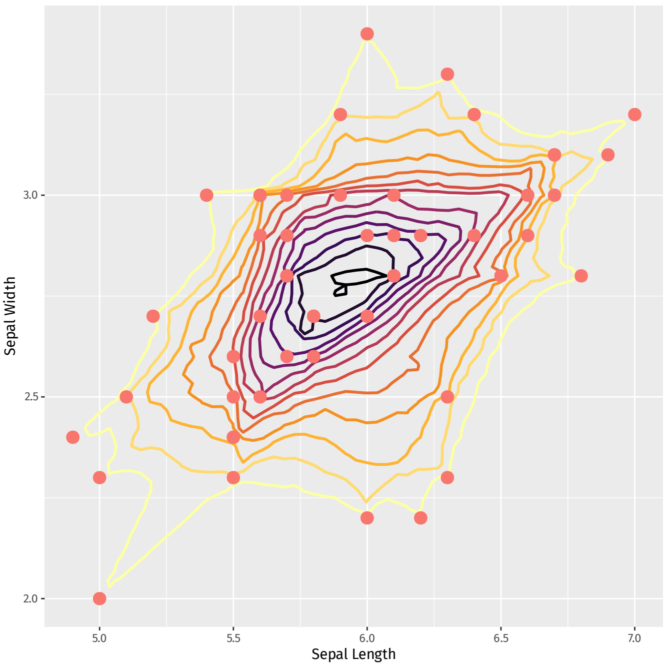


3.5
3.0
2.5
2.0
Sepal Width
5.0
5.5
6.0
6.5
7.0
Sepal Length

Simplicial depth

Examine the probability of $\boldsymbol{x}$ being contained in a random simplex.

$$D_S(x, F) = P(x \in \text{simplex}[X_1, \ldots, X_{p+1}] \mid X_i \overset{iid}{\sim} F).$$



Sepal Width
3.0
2.5
2.0
5.0
5.5
6.0
6.5
7.0
Sepal Length

Projection depth

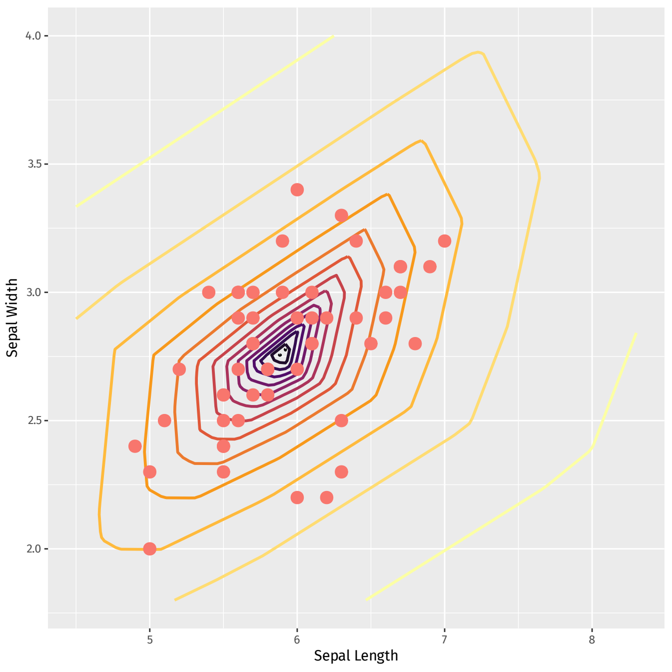
Examine the maximum outlyingness of $\boldsymbol{x}$ with respect to projections.

$$D_P(x, F) = \left(1 + \sup_{\|v\|=1} \frac{v^\top x - \mu(v^\top X)}{\sigma(v^\top X)}\right)^{-1}, \quad X \sim F.$$

A robust version can be defined as

$$D_P^*(x, F) = \left(1 + \sup_{\|v\|=1} \frac{v^\top x - \mathrm{median}(v^\top X)}{\mathrm{MAD}(v^\top X)}\right)^{-1}, \quad X \sim F,$$

$$\mathrm{MAD}(Y) = \mathrm{median}(|Y - \mathrm{median}(Y)|).$$


4.0
3.5
3.0
2.5
2.0
Sepal Width
5
6
7
8
Sepal Length

Why not use likelihood contours?

The ‘Curse of Dimensionality’.

Why not use likelihood contours?

The ‘Curse of Dimensionality’.

Additionally, consider a uniform distribution, say on a unit ball. This has non-trivial depth contours, but no proper density contours.

The Depth-Depth plot

Depth-Depth plots

Let F, G be two distributions on $\mathbb{R}^p$, and let D be a depth function. We construct the D-D plot

$$DD(F, G) = \{(D(\mathbf{x}, F),\ D(\mathbf{x}, G)) : \mathbf{x} \in \mathbb{R}^p\}.$$

Given data $\mathscr{D}_F, \mathscr{D}_G$, we may instead look at the D-D plot

$$DD(\hat{F}_m, \hat{G}_n) = \left\{\left(D(\mathbf{x}, \hat{F}_m),\ D(\mathbf{x}, \hat{G}_n)\right) : \mathbf{x} \in \mathscr{D}_F \cup \mathscr{D}_G\right\}.$$

Liu, R.Y., Parelius, J.M, & Singh, K. (1999) Multivariate analysis by data depth: descriptive statistics, graphics and inference

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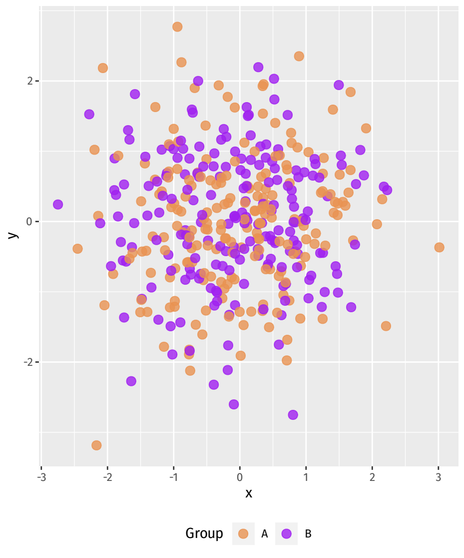
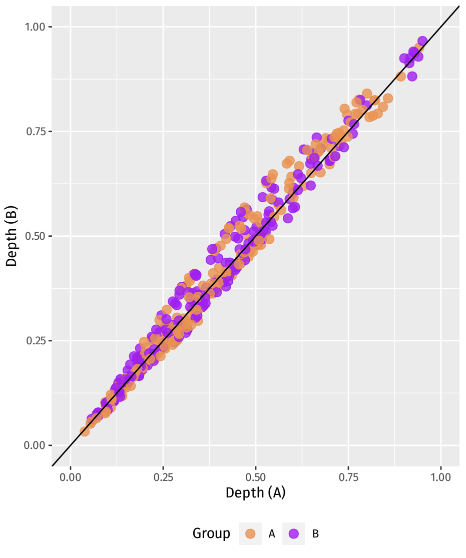
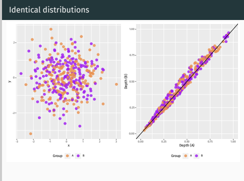
Liu, R.Y., Parelius, J.M, & Singh, K. (1999) Multivariate analysis by data depth: descriptive statistics, graphics and inference

- The area of $DD(F, G)$ can serve as an affine-invariant measure of the discrepancy between F and G.
- The D-D plot gives an ℓ-variate visualization of ℓ groups of data regardless of what the original data looks like (multivariate, functional).

Identical distributions

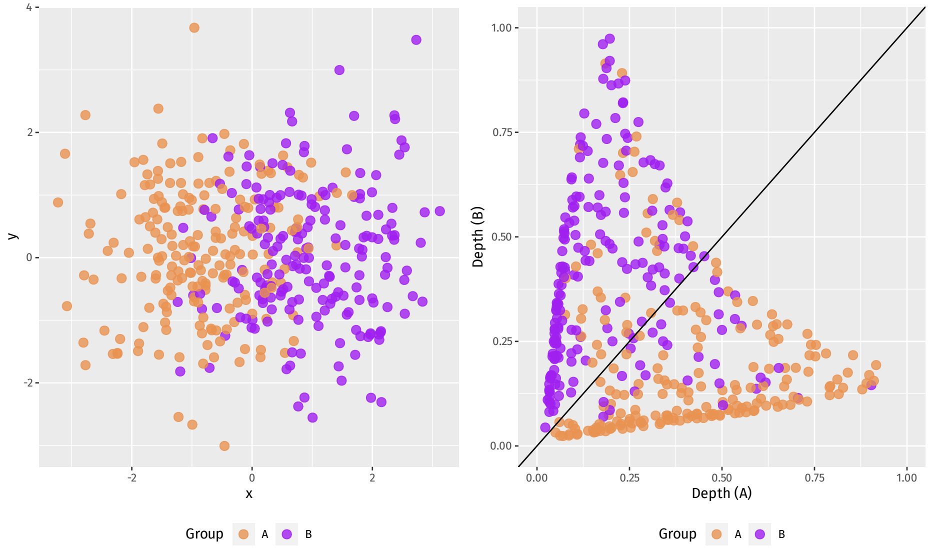




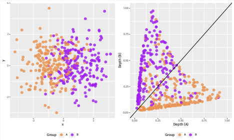


Both groups from standard normal distributions.

Location difference

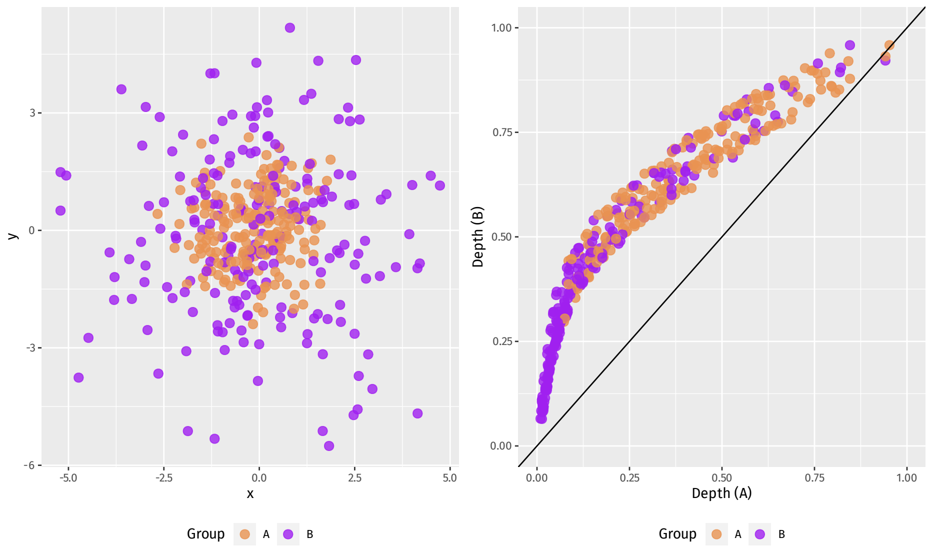


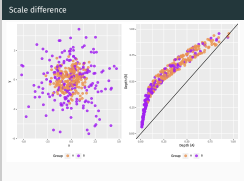
Location difference



Means shifted to $(-1, 0)^\top$ and $(1, 0)^\top$.

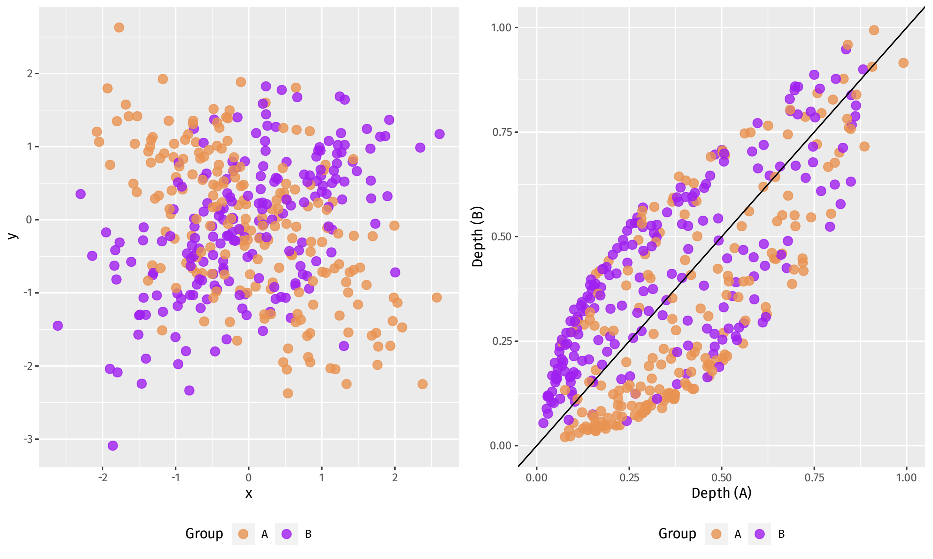
Scale difference

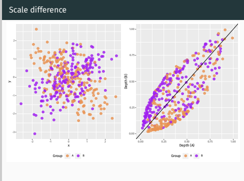




Covariances $\mathbb{I}_2$ and $4\mathbb{I}_2$.

Scale difference

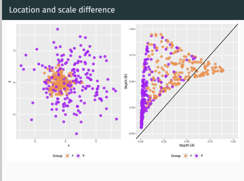




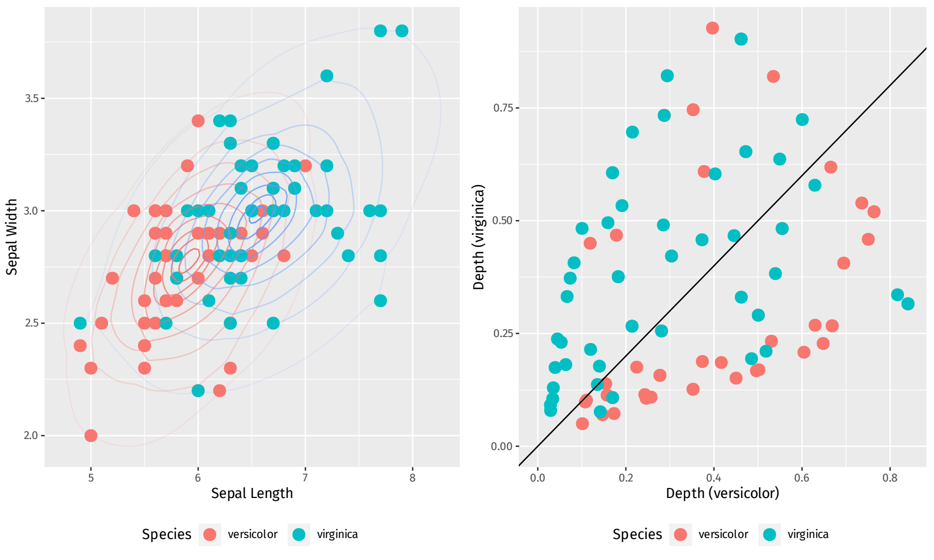
Covariances

$$\begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}.$$

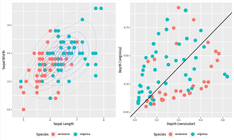
Location and scale difference



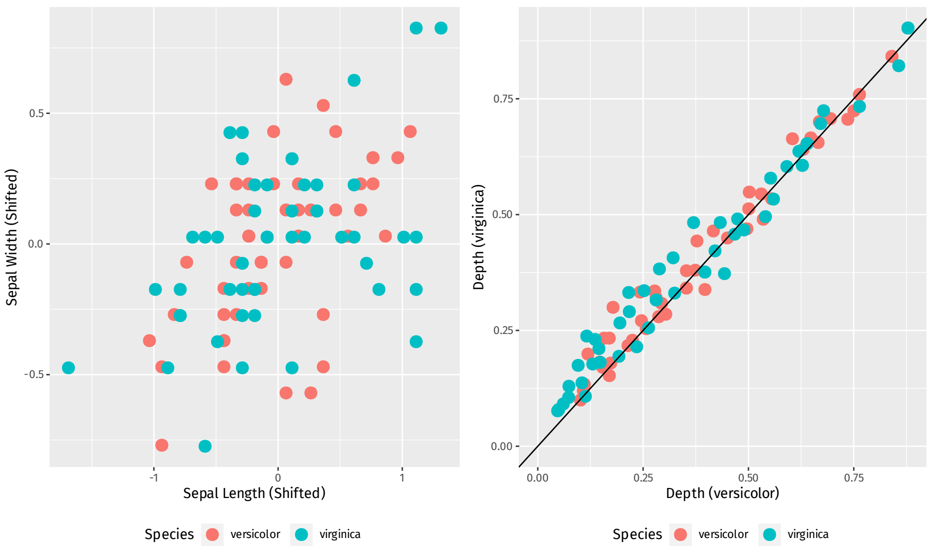
- Means $(-1, 0)^\top$ and $(1, 0)^\top$.
- Covariances $\mathbb{I}_2$ and $9\mathbb{I}_2$.



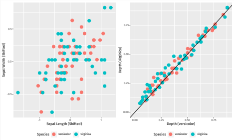
3.5
3.0
2.5
2.0
Sepal Width
5
6
7
8
Sepal Length
Species
versicolor
virginica
0.75
0.50
0.25
0.00
Depth (virginica)
0.0
0.2
0.4
0.6
0.8
Depth (versicolor)
Species
versicolor
virginica



D-D plot indicative of location difference.



Sepal Width (Shifted)
0.5
0.0
-0.5
-1
0
1
Sepal Length (Shifted)
Species
versicolor
virginica
Depth (virginica)
0.75
0.50
0.25
0.00
0.00
0.25
0.50
0.75
Depth (versicolor)
Species
versicolor
virginica



Data has been shifted so that the locations coincide.

Depth based classification

Maximum depth classifiers

Given a point $\mathbf{x} \in \mathbb{R}^p$, assign it to the class with respect to which it has maximum depth. In other words, choose

$$\hat{k}(\mathbf{x}) = \arg\max_j D(\mathbf{x}, \hat{F}_j).$$

Under certain conditions, this asymptotically performs on par with the Bayes classifier.

Ghosh, A.K., & Chaudhuri, P. (2005) On maximum depth and related classifiers

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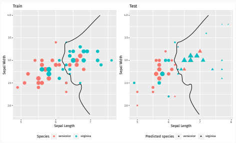
Maximum depth classification corresponds to using the $x = y$ line to separate points in the D-D plot.


Train
Test
Sepal Width
Sepal Length
4.0
3.5
3.0
2.5
2.0
5
6
7
8
Species
versicolor
virginica
Predicted species
versicolor
virginica



- This figure illustrates maximum depth classification on the same multivariate data shown earlier, using spatial depth.
- The depth contours are learned from training data.
- The black curve denotes the learned decision boundary.
- Classification accuracies hover around 70%.

Relative data depth

The *relative data depth*

$$\mathrm{ReD}(\boldsymbol{x}) = D(\boldsymbol{x}, \hat{F}_{\hat{k}(\boldsymbol{x})}) - \max_{j \neq \hat{k}(\boldsymbol{x})} D(\boldsymbol{x}, \hat{F}_j)$$

gives a measure of confidence in the classification of $\boldsymbol{x}$.

Jörnsten, R. (2004) Clustering and classification based on the L_1 data depth

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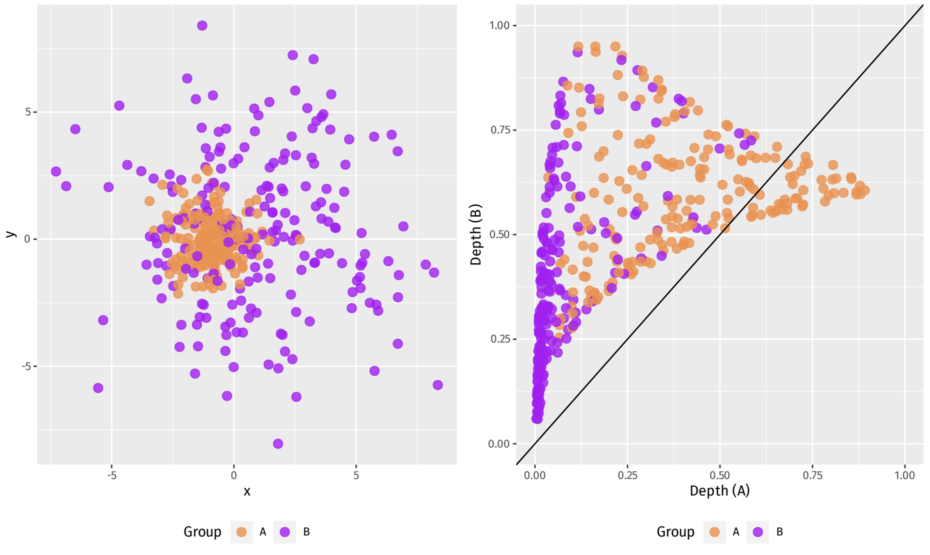
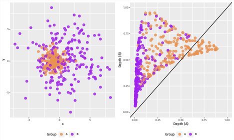
- This can be used to identify and remove ‘noisy’ examples from the training set.
- This can also be used as a measure of dissimilarity in clustering, with an objective function

$$\frac{1}{N} \sum_{k} \sum_{\boldsymbol{x}_i \in C(k)} \mathrm{ReD}(\boldsymbol{x}_i).$$



y
x
5
0
-5
Group
A
B
Depth (B)
Depth (A)
1.00
0.75
0.50
0.25
0.00



This illustrates that the maximum depth classifier may not always be appropriate.

Depth-Depth classifiers

Given data $\mathscr{D}_F, \mathscr{D}_G$, look at the D-D plot

$$DD(\hat{F}_m, \hat{G}_n) = \left\{ \Big(D(\mathbf{x}_i, \hat{F}_m),\ D(\mathbf{x}_i, \hat{G}_n) \Big) : \mathbf{x}_i \in \mathscr{D}_F \cup \mathscr{D}_G \right\},$$

and find a function ϕ which separates points from the two classes.

For $\mathbf{x} \in \mathbb{R}^p$, check which region the point $(D(\mathbf{x}, \hat{F}_m),\ D(\mathbf{x}, \hat{G}_n))$ lies in, and assign it to the corresponding class.

Li, J., Cuestas-Albertos, J.A., & Liu, R.Y. (2012) DD-Classifier: Nonparametric Classification Procedure Based on DD-Plot

Depth-Depth classifiers

Given data $\mathscr{D}_F, \mathscr{D}_G$, look at the D-D plot

$$DD(\hat{F}_m, \hat{G}_n) = \left\{ \left(D(x_i, \hat{F}_m),\, D(x_i, \hat{G}_n) \right) : x_i \in \mathscr{D}_F \cup \mathscr{D}_G \right\},$$

and find a function ϕ which separates points from the two classes.

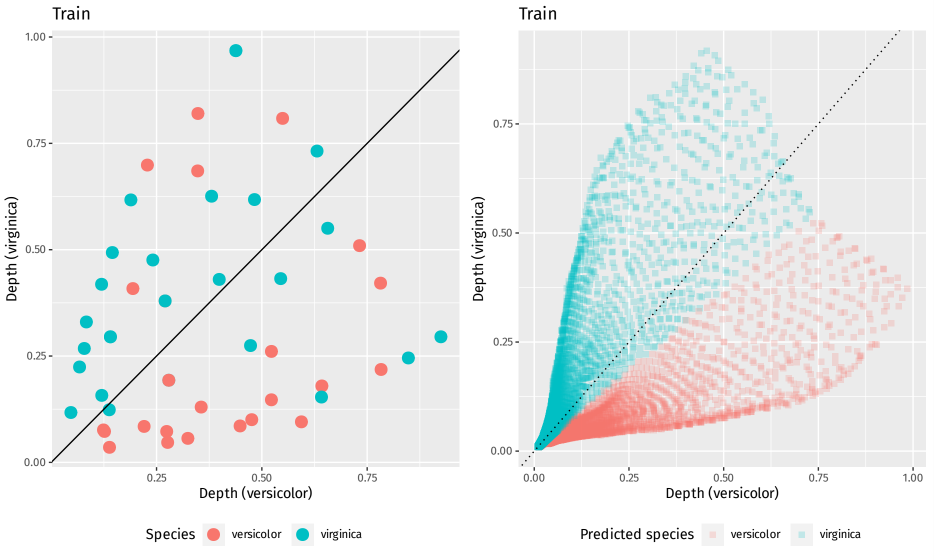
For $x \in \mathbb{R}^p$, check which region the point $(D(x, \hat{F}_m),\, D(x, \hat{G}_n))$ lies in, and assign it to the corresponding class.

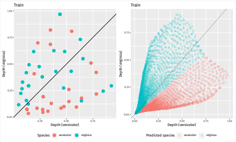
Li, J., Cuestas-Albertos, J.A., & Liu, R.Y. (2012) DD-Classifier: Nonparametric Classification Procedure Based on DD-Plot

- The D-D plot converts the ℓ-class classification problem to one in a ℓ-variate setting, regardless of what the original data looks like (multivariate, functional).
- The separating function ϕ is approximated by searching in a class of functions Γ, for instance, the family of increasing functions, or the family of polynomials.
- The two class DD classifier is easily extended to ℓ groups, in the form of the DD^G classifier. The data transformed via

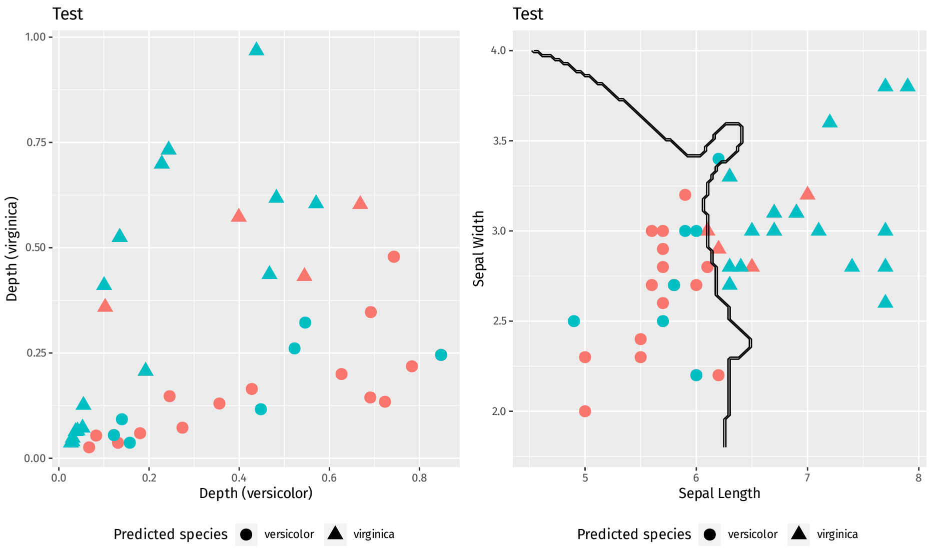
$$x \mapsto (D(x, \hat{F}_1), \ldots, D(x, \hat{F}_\ell))$$

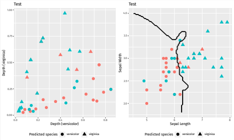
  can be classified using any existing multivariate classifier (LDA, kNN, GLM, etc).


Train
Depth (virginica)
Depth (versicolor)
1.00
0.75
0.50
0.25
0.00
0.25
0.50
0.75
Species
versicolor
virginica
Train
Depth (virginica)
Depth (versicolor)
0.75
0.50
0.25
0.00
0.00
0.25
0.50
0.75
1.00
Predicted species
versicolor
virginica



- The figure on the left shows the D-D plot for the training data.
- The figure on the right shows the locations of points (originally taken from a grid in the real data space) in the D-D plot. They are coloured according to the class predicted by the DD classifier, using polynomial boundaries.
- In this instance, the classification rule agrees closely with the maximum depth classifier rule. This is illustrated by the decision boundary in the D-D plot almost coinciding with the diagonal.


Test
1.00
0.75
0.50
0.25
0.00
Depth (virginica)
0.0
0.2
0.4
0.6
0.8
Depth (versicolor)
Predicted species
versicolor
virginica
Test
4.0
3.5
3.0
2.5
2.0
Sepal Width
5
6
7
8
Sepal Length
Predicted species
versicolor
virginica



- The figure on the left shows the predictions for the testing data on the D-D plot.
- The figure on the right shows the predictions for the testing data in the original space.
- The black curve denotes the decision boundary.
- Classification accuracies hover aruond 70%.

Elliptic distributions

Suppose that the underlying population distributions are elliptic, i.e. their density functions are of the form

$$c_i|\Sigma_i|^{-1/2}\, h_i\left((x-\mu_i)^\top \Sigma_i^{-1}(x-\mu_i)\right)$$

for strictly decreasing functions h_i. Then, the *Mahalanobis*, *simplicial*, and *projection* depths $D(\cdot, F_i)$ are strictly increasing functions of the respective densities.

Thus, the Bayes rule involves comparing $\phi(D(x,F))$ and $D(x,G)$ for some strictly increasing function ϕ.

Li, J., Cuestas-Albertos, J.A., & Liu, R.Y. (2012) DD-Classifier: Nonparametric Classification Procedure Based on DD-Plot

Depth functions for Functional Data

Integrated, infimal, and random projection depths

$$D_{int}(X, F_X) = \int_T D(X(t), F_{X(t)})\, w(t)\, dt.$$

$$D_{inf}(X, F_X) = \inf_{t \in T} D(X(t), F_{X(t)}).$$

$$D_{RP}(X, F_X) = \inf_{\phi} D(\langle X, \phi \rangle, F_{\langle X, \phi \rangle}).$$

Gijbels, I., & Nagy, S. (2017) On a General Definition of Depth for Functional Data

Outlyingness matrices

Given a random p-variate function X, define a pointwise outlyingness function as

$$\mathbf{O}(X(t), F_{X(t)}) = \left[\frac{1}{D(X(t), F_{X(t)})} - 1\right] \cdot \mathbf{v}(t).$$

With this, define

$$\mathbf{MO}(X, F_X) = \int_T \mathbf{O}(X(t), F_{X(t)})\, w(t)\, dt,$$

$$VO(X, F_X) = \int_T \|\mathbf{O}(X(t), F_{X(t)}) - \mathbf{MO}(X, F_X)\|^2\, w(t)\, dt.$$

Dai, W., & Genton, M.G. (2018) An outlyingness matrix for multivariate functional data classification

Outlyingness matrices

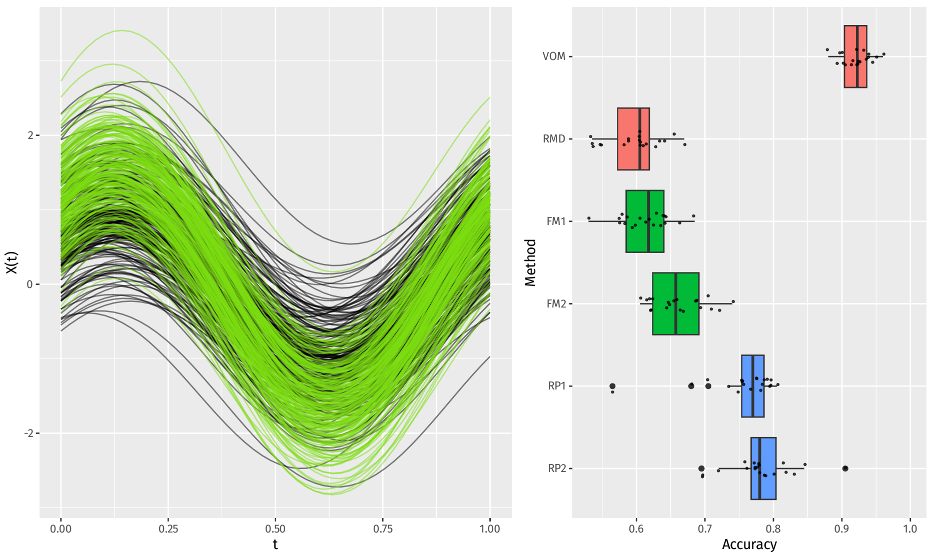
Furthermore, denoting

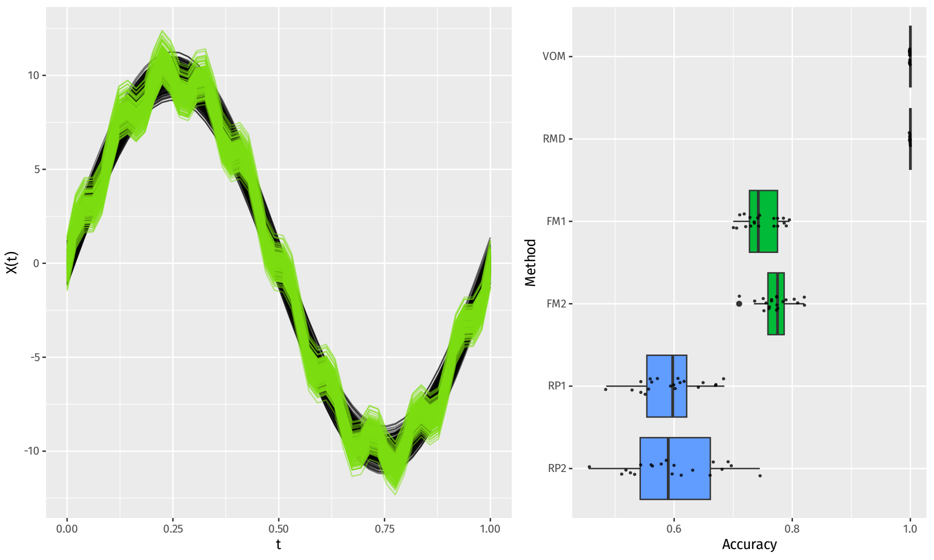
$$\tilde{\mathbf{O}}(X(t), F_{X(t)}) = \mathbf{O}(X(t), F_{X(t)}) - \mathbf{MO}(X, F_X),$$

define the *variational outlyingness matrix*

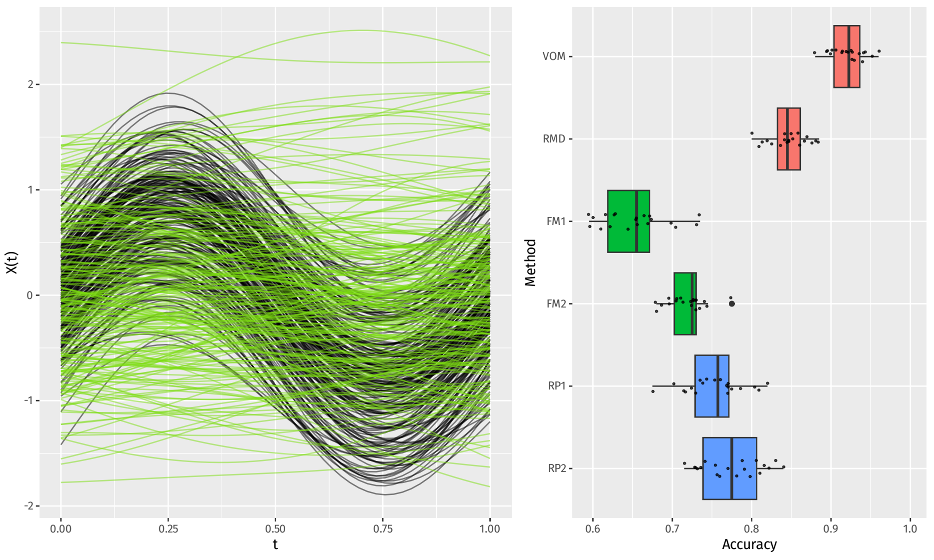
$$VOM(X, F_X) = \int_T \tilde{\mathbf{O}}(X(t), F_{X(t)})\, \tilde{\mathbf{O}}(X(t), F_{X(t)})^\top w(t)\, dt.$$

Use either the feature vector $(\mathbf{MO}^\top, VO)^\top$ or $\|VOM\|$ for classification.


X(t)
2
0
-2
0.00
0.25
0.50
0.75
1.00
t
Method
VOM
RMD
FM1
FM2
RP1
RP2
0.6
0.7
0.8
0.9
1.0
Accuracy

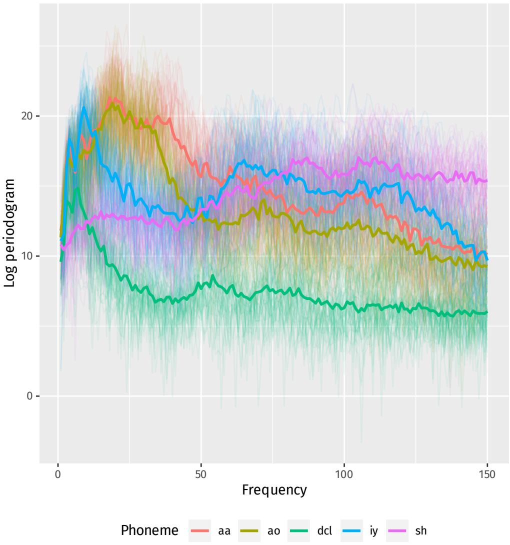


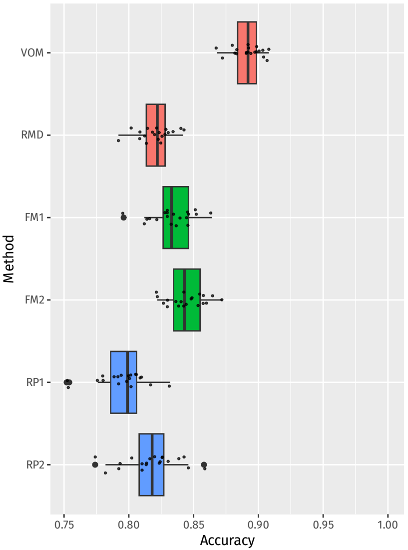
X(t)
10
5
0
-5
-10
0.00
0.25
0.50
0.75
1.00
t
Method
VOM
RMD
FM1
FM2
RP1
RP2
0.6
0.8
1.0
Accuracy

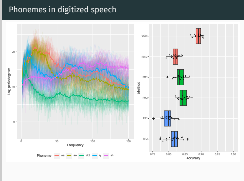


X(t)
2
1
0
-1
-2
0.00
0.25
0.50
0.75
1.00
t
Method
VOM
RMD
FM1
FM2
RP1
RP2
0.6
0.7
0.8
0.9
1.0
Accuracy

Phonemes in digitized speech







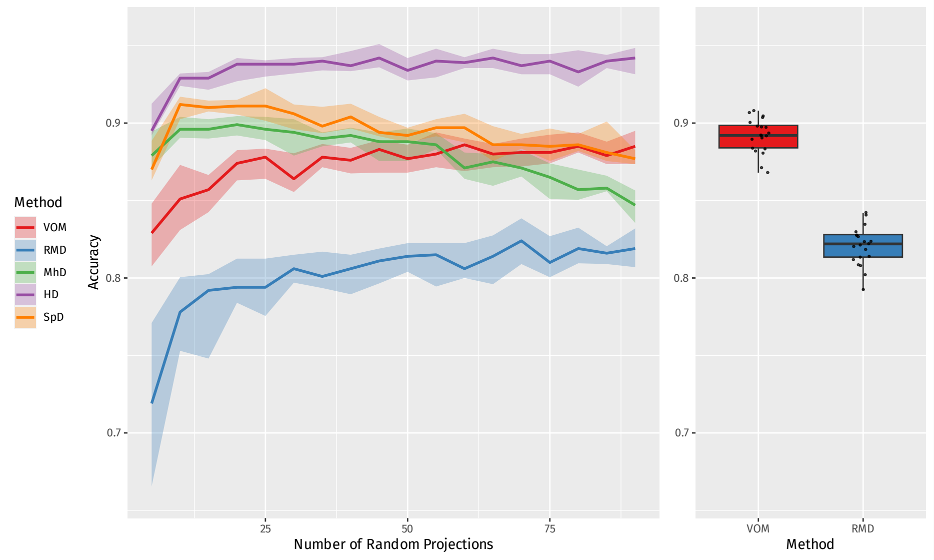
- This figure illustrated periodograms obtained from digitized speech.
- Different groups correspond to the pronunciation of different phonemes.
- The thicker lines denote the median curves from the corresponding group.
- This data is available as ‘phoneme data’ from the `fds` package in R.

Functional $\to$ Multivariate, via random projections

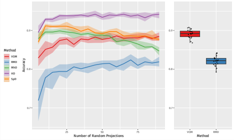
Replace $\{X(t)\}_{t\in T}$ with $\{\langle X, \phi_j\rangle\}_{j=1}^{\ell}$, where $\phi_1, \ldots, \phi_\ell$ are random functions and

$$\langle X, \phi\rangle = \int_T \langle X(t), \phi(t)\rangle \, w(t)\, dt.$$

Phonemes in digitized speech revisited



Phonemes in digitized speech revisited



- The random functions $\phi_1, \dots, \phi_\ell$ have been generated by a Gaussian process with an exponential covariance kernel.
- The last three methods employ the maximum depth classifier (with the corresponding depths), applied on the transformed data

$$X \mapsto \left(\langle X, \phi_1 \rangle, \dots, \langle X, \phi_\ell \rangle\right).$$

- The degradation in performance of the Mahalanobis classifier is likely due to the worsening estimate of the covariance matrix as the number of projections (hence the dimension) ℓ increases.

Do depth functions completely characterize probability distributions?

Sometimes!

Do depth functions completely characterize probability distributions?

Sometimes!

This has implications in the consistency of depth based tests and classifiers, where all information about the given data/distribution is obtained via depth.

Halfspace depth revisited

The halfspace depth characterizes discrete probability distributions, i.e. if $D_H(\cdot, P) = D_H(\cdot, Q)$ and one of P, Q is discrete, then $P = Q$.

The halfspace depth also characterizes elliptic probability distributions.

Cuesta-Albertos, J.A., & Nieto-Reyes, A. (2008) The Tukey and the random Tukey depths characterize discrete distributions

Kong, L., & Zuo, Y. (2010) Smooth depth contours characterize the underlying distribution

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The halfspace depth characterizes distributions P in $\mathbb{R}^p$ with contiguous support such that the depth contours for $0 < p < 1/2$ are *smooth* and the *maximal mass of P at a hyperplane*

$$\Delta(P) = \sup P(v^\top X = c) = 0.$$

A counterexample

Consider $X \sim P$, $Y \sim Q$ where

$$\psi_X(\boldsymbol{t}) = \exp(-\|\boldsymbol{t}\|_1^{1/2}), \qquad \psi_Y(\boldsymbol{t}) = \exp(-\|\boldsymbol{t}\|_{1/2}^{1/2}).$$

Observe that the *marginals* of X and Y are identically distributed!

This is because they have the same characteristic function,

$$\psi(t) = \exp(-|t|^{1/2}).$$

Nagy, S. (2021) Halfspace depth does not characterize probability distributions

A counterexample

Next, if $\psi_Z(\boldsymbol{t}) = \psi(\|\boldsymbol{t}\|_\alpha)$, then $\boldsymbol{v}^\top \boldsymbol{Z} \stackrel{d}{=} \|\boldsymbol{v}\|_\alpha Z_1$. Such distributions are called α-*symmetric*.

Using this, it can be shown that

$$D_H(\boldsymbol{x}, P) = D_H(\boldsymbol{x}, Q) = F(-\|\boldsymbol{x}\|_\infty),$$

where F is the cdf of X_1.

A counterexample

Next, if $\psi_Z(\boldsymbol{t}) = \psi(\|\boldsymbol{t}\|_\alpha)$, then $\boldsymbol{v}^\top \boldsymbol{Z} \overset{d}{=} \|\boldsymbol{v}\|_\alpha Z_1$. Such distributions are called *α-symmetric*.

Using this, it can be shown that

$$D_H(\boldsymbol{x}, P) = D_H(\boldsymbol{x}, Q) = F(-\|\boldsymbol{x}\|_\infty),$$

where F is the cdf of X_1.

- Observe that

$$\begin{aligned} D_H(\boldsymbol{x}, F_Z) &= \inf_{\boldsymbol{v} \neq 0} P(\boldsymbol{v}^\top Z \leq \boldsymbol{v}^\top \boldsymbol{x}) \\ &= \inf_{\boldsymbol{v} \neq 0} P\left(Z_1 \leq \frac{\boldsymbol{v}^\top \boldsymbol{x}}{\|\boldsymbol{v}\|_\alpha}\right) \\ &= P\left(Z_1 \leq \inf_{\|\boldsymbol{v}\|_\alpha = 1} \boldsymbol{v}^\top \boldsymbol{x}\right). \end{aligned}$$

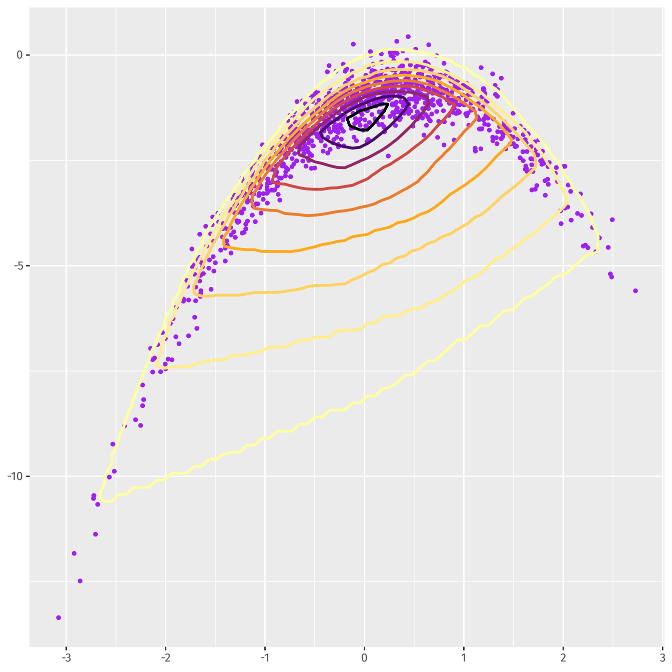
- The infimum $-\|\boldsymbol{x}\|_\infty$ is achieved when $\boldsymbol{v} = \boldsymbol{e}_j$.
- This is easy to see when $\alpha = 1$ (optimization over a convex hull). When $0 < \alpha \leq 1$, use $\|\boldsymbol{v}\|_\alpha \geq \|\boldsymbol{v}\|_1$.

Future work

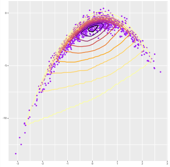
Local depths

The notions of depth discussed so far work well with elliptic, unimodal distributions, but fail to capture the natures of more general distributions.

Agostinelli, C., & Romanazzi, M. (2011) Local depth



0
-5
-10
-3
-2
-1
0
1
2
3



Halfspace depth contours of data drawn from a ‘banana’ shaped distribution, generated by first drawing

$$X \sim \mathcal{N}(0, \Sigma), \qquad \Sigma = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix},$$

then setting

$$Y = \begin{bmatrix} aX_1 \\ X_2/a + b((aX_1)^2 + a^2) \end{bmatrix}, \qquad a = 1, \quad b = 1.$$

Distribution-free procedures

Use ideas from optimal transportation to investigate more canonical notions of depth (for instance, the Monge-Kantorovich depth), and thereby establish procedures independent of the underlying distributions/spaces.

Chernozhukov, V., Galichon, A., Hallin, M., & Henry, M. (2017) Monge–Kantorovich depth, quantiles, ranks and signs

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