Notes from a course* on

Representation Theory of Finite Groups

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1. Linear representations of groups

Definition 1.1 (Linear representation): Let G be a finite group, and let V be a vector space. A linear representation (σ, V) of G is a homomorphism

$$\sigma : G \to \mathrm{GL}(V).$$

Example 1.1.1: The *trivial* representation of G is defined by $g \mapsto \mathrm{id}_V$.

Example 1.1.2: Consider a vector space V of dimension $\mathrm{ord}(G)$, and pick a basis $\{e_h\}_{h \in G}$. The *regular* representation $\tau : G \to \mathrm{GL}(V)$ of G is defined as follows: $\tau(g)$ sends each of the basis vectors $e_h \mapsto e_{gh}$.

The following propositions show that it is possible to define group representations in terms of a special class of group actions of G on the vector space V.

Proposition 1.2: Let G be a finite group, and let V be a vector space. Let $\rho : G \times V \to V$ be a group action of G on V, such that each for each G, the map $v \mapsto \rho(g, v)$ is linear. Then, (σ, V) is a linear representation, where

$$\sigma : G \to \mathrm{GL}(V), \qquad g \mapsto (v \mapsto \rho(g, v)).$$

Proposition 1.3: Let (σ, V) be a linear representation. Then, the map

$$\rho : G \times V \to V, \qquad (g, v) \mapsto \sigma(g)(v)$$

is a group action of G on V, where for each $g \in G$, the map $v \mapsto \rho(g, v)$ is linear.

In this discussion, we will always work with finite groups, as well as finite dimensional vector spaces over a base field K. Typically, we will consider $K = \mathbb{C}$.

We will often abbreviate (σ, V) with V, and $\sigma(g)$ with g when the presence of σ is clear from context.

Definition 1.4: The dimension of a representation (σ, V) is $\dim(V)$.

Example 1.4.1: The only one dimensional representation of S_3 in $\mathbb{C}^\times$ is the sign homomorphism. To see this, consider an arbitrary homomorphism $\sigma : S_3 \to \mathbb{C}^\times$. Note that $\ker(\sigma)$ must be a normal subgroup of S_3, hence must be one of $\{e\}, A_3, S_3$. The third option yields the trivial representation $\sigma = \mathrm{id}_{\mathbb{C}^\times}$, and the first option gives the contradiction $S_3 \cong \mathrm{im}(\sigma) \subset \mathbb{C}^\times$ (the right side is abelian while the left is not). This leaves $\ker(\sigma) = A_3$, i.e. $\sigma(g) = 1$ for all even permutations $g \in S_3$. The remaining elements of S_3 (the odd permutations) must be sent to -1, since for any odd permutation $h \in S_3$, the permutation h^2 is even, so $\sigma(h)^2 = \sigma(h^2) = 1$. The result is precisely the sign homomorphism

$$\sigma : S_3 \to \mathbb{C}^\times, \qquad g \mapsto \begin{cases} +1 \text{ if } g \in A_3 \\ -1 \text{ if } g \notin A_3. \end{cases}$$

Example 1.4.2: Construct an equilateral triangle in $\mathbb{C}^2$ centered at the origin, and consider the natural action of S_3 on it (permuting its vertices v_1, v_2, v_3). This induces a two dimensional representation $\sigma : S_3 \to \mathrm{GL}(\mathbb{C}^2)$. Note that $\{v_1, v_2\}$ forms a basis of $\mathbb{C}^2$; the third vertex can be obtained via $v_3 = -v_1 - v_2$. With this, we can calculate the image of (v_1, v_2) under the action of each $g \in S_3$, and hence the matrices of $\sigma(g)$ in the given basis as follows.

g	$(\sigma(g)(v_1), \sigma(g)(v_2))$	Matrix of $\sigma(g)$
e	(v_1, v_2)	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
(12)	(v_2, v_1)	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
(23)	$(v_1, v_3) = (v_1, -v_1 - v_2)$	$\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$
(31)	$(v_3, v_2) = (-v_1 - v_2, v_2)$	$\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$
(123)	$(v_2, v_3) = (v_2, -v_1 - v_2)$	$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$
(321)	$(v_3, v_1) = (-v_1 - v_2, v_1)$	$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

Consider the setting $K = \mathbb{C}$. The fact that G is a finite group means that each element $g \in G$ has finite order, hence satisfies $g^m = 1$ for some $m \mid \mathrm{ord}(G)$. This means that $\sigma(g)^m = \mathrm{id}_V$, whence $x^m - 1$ is an annihilating polynomial for $\sigma(g)$. A consequence of this is that the minimal polynomial of $\sigma(g)$ is a factor of $x^m - 1$; but the latter splits into distinct linear factors. Furthermore, all eigenvalues of $\sigma(g)$ are roots of its minimal polynomial. This yields the following result.

Proposition 1.5: Suppose that $K = \mathbb{C}$. Let (σ, V) be a representation of G, and let $g \in G$. Then, $\sigma(g)$ is diagonalizable, and its eigenvalues are roots of unity.

2. Subrepresentations

Definition 2.1 (Stable subspace): Let (σ, V) be a representation of G, and let $W \subseteq V$ be a subspace of V. We say that W is a stable subspace of V if it is invariant under the action of G, i.e. $gw \in W$ for all $g \in G$, $w \in W$.

Example 2.1.1: Let S_3 act on $\mathbb{C}^3$ by permuting the basis vectors $\{e_1, e_2, e_3\}$. Then, the subspace $\mathrm{span}\{e_1 + e_2 + e_3\}$ is stable.

Definition 2.2 (Subrepresentation): Let W be a stable subspace of V. We say that (σ, W) is a subrepresentation of (σ, V).

Theorem 2.3: Suppose that $\operatorname{char}(K) \nmid \operatorname{ord}(G)$. Let W be a stable subspace of V. Then, there exists a stable subspace W' of V such that $V = W \oplus W'$.

When working with the field $K = \mathbb{C}$, Theorem 2.3 admits a simpler form by invoking the orthocomplement of $W \subseteq V$, with respect to a suitable Hermitian form on V. We say that a Hermitian form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ is G-invariant if for all $g \in G$, $v, v' \in V$, we have $\langle gv, gv' \rangle = \langle v, v' \rangle$.

Theorem 2.4: Suppose that $K = \mathbb{C}$. If W is a stable subspace of V, then $W^\perp$ is a stable subspace of V, with $V = W \oplus W^\perp$.

Remark: The subspace $W^\perp$ is defined with respect to a non-degenerate G-invariant Hermitian form.

Proof: For all $g \in G$, $w \in W$, $w' \in W^\perp$, observe that $g^{-1}w \in W$, so

$$\langle gw', w \rangle = \langle gw', gg^{-1}w \rangle = \langle w', g^{-1}w \rangle = 0,$$

whence $gw' \in W^\perp$. □

Example 2.4.1: Continuing Example 2.1.1, the subspace $\operatorname{span}\{e_1 - e_2, e_2 - e_3, e_3 - e_1\}$ is also stable under the action of S_3. This gives a two dimensional subrepresentation of S_3. In fact, it is easy to check that the matrices describing this representation in the basis $\{2e_1 - e_2 - e_3, 2e_2 - e_3 - e_1\}$ are precisely the same as those in Example 1.4.2, making these two representations identical in some sense.

Remark: Given any Hermitian form $\langle \cdot, \cdot \rangle : V \times V \to C$, we can obtain a G-invariant Hermitian form on V defined by

$$(u, v) \mapsto \sum_{g \in G} \langle gu, gv \rangle.$$

Returning to Theorem 2.3, observe that if π_W is a projection onto the subspace W, then we may write $V = W \oplus \ker(\pi_W)$. With this in mind, we will construct the required subspace W' as the kernel of a suitable projection map π_W. For this, we demand that π_W be G-invariant.

Definition 2.5: A linear map $f : V \to V'$ is called G-invariant if it is compatible with the G-action, i.e. for all $g \in G$, $v \in V$, we have $f(gv) = gf(v)$.

Note that the above definition implicitly deals with *two* representations (σ, V) and (σ', V) of G. The indicated property really looks like $\sigma'(g)(f(v)) = f(\sigma(g)(v))$ when written in full.

Lemma 2.6: Let $f : V \to V'$ be G-invariant. Then,

1. $\ker(f)$ is a stable subspace of V.
2. $\operatorname{im}(f)$ is a stable subspace of V'.

Given any linear map $f : V \to V'$, we can construct a G-invariant linear map via

$$\tilde{f} : V \to V', \qquad v \mapsto \sum_{g \in G} g f(g^{-1} v).$$

With this, we are ready to furnish a proof of our theorem.

Proof of Theorem 2.3: Let $\pi : V \to W$ be any projection onto W. Observe that

$$\pi_W : V \to W, \qquad v \mapsto \frac{1}{\operatorname{ord}(G)} \sum_{g \in G} g\pi(g^{-1} v)$$

is a G-invariant projection onto W. Setting $W' = \ker(\pi_W)$ completes the proof. □

Remark: Note how the assumption that $\operatorname{char}(\mathrm{K}) \nmid \operatorname{ord}(G)$ is crucial for defining the projection π_W.

3. Irreducible representations

Definition 3.1 (Irreducible representations): We way that a representation is irreducible if it admits no proper non-trivial subrepresentations.

In other words, a representation (σ, V) is irreducible *if and only if* the only G-invariant subspaces of V are $\{0\}, V$.

Example 3.1.1: All one dimensional representations are irreducible.

Theorem 3.2 (Maschke's Theorem): Suppose that $\operatorname{char}(K) \nmid \operatorname{ord}(G)$. Then, every representation of G over the field K can be written as a direct sum of irreducible representations of G.

Proof: Follows immediately from Theorem 2.3. □

Example 3.2.1: Combining Examples 2.1.1 and 2.4.1, we have the decomposition

$$\mathbb{C}^3 = \operatorname{span}\{e_1 + e_2 + e_3\} \oplus \operatorname{span}\{e_1 - e_2, e_2 - e_3, e_3 - e_1\}$$

into irreducible subrepresentations of S_3.

When we say that two representations (σ, V) and (σ', V') are isomorphic, denoted $V \cong V'$, we mean that there exists a G-invariant linear bijection $f : V \to V'$. The following result offers a very powerful characterization of G-invariant maps between irreducible representations.

Theorem 3.3 (Schur's Lemma): Let V, V' be two irreducible representations of G, and let $f : V \to V'$ be a G-invariant linear map.

1. If $V \ncong V'$, then $f = 0$.
2. If $V = V'$ and K is algebraically closed, then f is a scalar map, i.e. $f = \lambda \operatorname{id}_V$ for some $\lambda \in K$.

Proof:

1. Suppose that $f \neq 0$. It suffices to show that f is an isomorphism; to do so, we make extensive use of Lemma 2.6.

First, $\ker(f) \subseteq V$ is stable, hence must be one of $\{0\}, V$ by the irreducibility of V. The assumption $f \neq 0$ forces $\ker(f) = \{0\}$, whence f is injective.

Next, $\operatorname{im}(f) \subseteq V'$ is stable, hence must be one of $\{0\}, V'$ by the irreducibility of V'. Again, $f \neq 0$ forces $\operatorname{im}(f) = V'$, whence f is surjective.

2. We have a G-invariant linear bijection $f : V \to V$; suppose that $f \neq 0$. Let λ be an eigenvalue of f, and observe that the map $(f - \lambda \ \mathrm{id}_V)$ is also G-invariant; indeed, for all $g \in G$, $v \in V$,

$$(f - \lambda)(gv) = f(gv) - \lambda gv = gf(v) - g\lambda v = g(f - \lambda)(v).$$

Since λ is an eigenvalue of f, we have $\ker(f - \lambda) \neq \{0\}$. Since $\ker(f - \lambda) \subseteq V$ is stable and V is irreducible, we must have $\ker(f - \lambda) = V$, whence $f - \lambda \, \mathrm{id}_V = 0$. □

Remark: Note how the existence of the scalar $\lambda \in K$ is guaranteed by the fact that K is algebraically closed.

Corollary 3.3.1: All $\mathbb{C}$-linear irreducible representations of finite abelian groups are one dimensional.

Proof: Let (σ, V) be an irreducible representation of a finite abelian group G. Check that for each $g \in G$, the linear map $\sigma(g) : V \to V$ is G-invariant, since it commutes with all $\sigma(h)$ for $h \in G$. From Schur's Lemma (Theorem 3.3), each $\sigma(g)$ is a scalar map. As a result, every one dimensional subspace of V is stable. The result now follows from the irreducibility of V. □

4. Characters

Definition 4.1 (Character): The character χ_V of a representation (σ, V) of G is the function

$$\chi_V : G \to K, \qquad g \mapsto \operatorname{tr}(\sigma(g)).$$

Example 4.1.1: $\chi_V(1) = \dim(V)$.

Observe that $\chi_V(g)$ is precisely the sum of eigenvalues of $\sigma(g)$. The eigenvalues of $\chi_V(g^{-1})$ are simply reciprocals of those of $\chi_V(g)$; in the setting $K = \mathbb{C}$, the following result is immediate from Proposition 1.5.

Proposition 4.2: Suppose that $K = \mathbb{C}$. Then, $\chi_V(g^{-1}) = \overline{\chi_V(g)}$.

The fact that the trace is invariant under conjugation, i.e. $\operatorname{tr}(tst^{-1}) = \operatorname{tr}(s)$, yields the following result.

Lemma 4.3: χ_V is a class function, i.e. χ_V is constant on conjugacy classes of G.

Lemma 4.4: Isomorphic representations have the same character.

Proof: Let $f : V \to V'$ be an isomorphism of representations (σ, V) and (σ', V') of G. Then for each $g \in G$, we have $f \circ \sigma(g) = \sigma'(g) \circ f$, hence $\sigma(g) = f^{-1} \circ \sigma'(g) \circ f$. Taking the trace of both sides and using the cyclic property gives $\operatorname{tr}(\sigma(g)) = \operatorname{tr}(\sigma'(g))$ as desired. □

4.1. Orthogonality

The space K^G of all maps $G \to K$ forms a vector space over K, with dimension $\operatorname{ord}(G)$. In the setting $K = \mathbb{C}$, we may define the following inner product.

$$\langle \cdot, \cdot \rangle : \mathbb{C}^G \times \mathbb{C}^G \to \mathbb{C}, \qquad (\varphi, \psi) \mapsto \frac{1}{\operatorname{ord}(G)} \sum_{g \in G} \varphi(g)\overline{\psi(g)}.$$

Remark: For characters χ, χ', Proposition 4.2 gives

$$\langle \chi, \chi' \rangle = \frac{1}{\operatorname{ord}(G)} \sum_{g \in G} \chi(g)\chi'(g^{-1}).$$

Theorem 4.5 (Orthogonality of characters): Suppose that $K = \mathbb{C}$. Let (σ, V), (σ', V') be two irreducible representations of G.

1. If $V \ncong V'$, then $\langle \chi_V, \chi_{V'} \rangle = 0$.
2. If $V \cong V'$, then $\langle \chi_V, \chi_{V'} \rangle = 1$.

Proof: Let $\{v_1, ..., v_n\}$ be a basis of V, and let $\{v_1', ..., v_m'\}$ be a basis of V'. Given any linear map $f : V \to V'$, we will denote $\tilde{f} = \sum_{g \in G} \sigma'(g) \circ f \circ \sigma(g)^{-1}$; recall that $\tilde{f}$ is G-invariant.

1. Observe that Schur's Lemma (Theorem 3.3) forces all such $\tilde{f} = 0$. In particular, consider the maps e_{ij} defined for each $1 \leq i \leq n$, $1 \leq j \leq m$ as

   $$e_{ij} : V \to V', \qquad \sum_i \alpha_i v_i \mapsto \alpha_i v_j'.$$

   These maps $\{e_{ij}\}$ form a basis of $\mathcal{L}(V, V')$. Check that the matrix entries obey

   $$[a \circ e_{ij} \circ b]_{k\ell} = [a]_{ki}[b]_{j\ell},$$

   so using $\tilde{e}_{ij} = 0$ gives the relations

   $$[\tilde{e}_{ij}]_{kl} = \sum_{g \in G} \left[\sigma'(g) \circ e_{ij} \circ \sigma(g)^{-1}\right]_{kl} = \sum_{g \in G} [\sigma'(g)]_{ki} \left[\sigma(g)^{-1}\right]_{j\ell} = 0$$

   for all $1 \leq i, k \leq n$, $1 \leq j, \ell \leq m$. These hold in particular for $i = k$, $j = \ell$; summing over $1 \leq i \leq n$, $1 \leq j \leq m$, we have

   $$\begin{aligned} 0 = \sum_{ij} \sum_{g \in G} [\sigma'(g)]_{ii} \left[\sigma(g)^{-1}\right]_{jj} &= \sum_{g \in G} \left(\left(\sum_i [\sigma'(g)]_{ii} \right) \left(\sum_j \left[\sigma(g)^{-1}\right]_{jj} \right) \right) \\ &= \sum_{g \in G} \chi_V(g)\chi_{V'}(g^{-1}) \\ &= \operatorname{ord}(G) \langle \chi_V, \chi_{V'} \rangle. \end{aligned}$$

2. Schur's Lemma (Theorem 3.3) forces all such $\tilde{f} = \lambda_f \operatorname{id}_V$ for scalars $\lambda_f \in \mathbb{C}$. To extract λ_f, take the trace of both sides to obtain

$$n\lambda_f = \dim(V)\lambda_f = \sum_{g\in G} \operatorname{tr}\left(\sigma'(g)\circ f\circ \sigma(g)^{-1}\right) = \operatorname{ord}(G)\operatorname{tr}(f).$$

With this, each $\tilde{e}_{ij} = \lambda_{ij}\delta_{ij}\operatorname{id}_V$, where $\lambda_{ij} = \operatorname{ord}(G)/n$. Thus, we obtain the relations

$$\sum_{g\in G} [\sigma'(g)]_{ki}\left[\sigma(g)^{-1}\right]_{j\ell} = \frac{1}{n}\operatorname{ord}(G)\delta_{ij}\delta_{kl}$$

for all $1 \le i, j, k, \ell \le n$. Following a similar process as before,

$$\begin{aligned}\operatorname{ord}(G)\langle\chi_V, \chi_{V'}\rangle &= \sum_{g\in G}\left(\left(\sum_i [\sigma'(g)]_{ii}\right)\left(\sum_j \left[\sigma(g)^{-1}\right]_{jj}\right)\right) \\ &= \sum_{ij}\sum_{g\in G}[\sigma'(g)]_{ii}\left[\sigma(g)^{-1}\right]_{jj} \\ &= \sum_{ij}\frac{1}{n}\operatorname{ord}(G)\delta_{ij} \\ &= \operatorname{ord}(G)\end{aligned}$$

This completes the proof. □

With this, the characters of irreducible representations form an orthonormal subset of class functions on G. To check whether a representation V is irreducible or not, it is enough to verify that $\langle\chi_V, \chi_V\rangle = 1$.

Corollary 4.5.1: The number of irreducible representations of G (up to isomorphism) is at most the number of conjugacy classes of G.

Given any representation V of G, we can use Maschke's Theorem (Theorem 3.2) to decompose it as a direct sum of (non-isomorphic) irreducible representations $V_1, ..., V_k$, with multiplicities $m_1, ..., m_k$. By representing the elements of G as matrices in block diagonal form, we can derive the following result.

Lemma 4.6: Let $V_1, ..., V_k$ be irreducible representations of G, and let

$$V \cong m_1V_1 \oplus \cdots \oplus m_kV_k.$$

Then,

$$\chi_V = m_1\chi_{V_1} + \cdots + m_k\chi_{V_k}.$$

The multiplicities can be recovered as $m_i = \langle\chi_V, \chi_{V_i}\rangle$.

This immediately tells us that $\chi_V = \chi_{V'}$ if and only if $V \cong V'$. Furthermore, we have the relation

$$\langle\chi_V, \chi_V\rangle = \sum_i m_i^2.$$

4.2. The character table for S_3

We have now established that the trivial representation, the one dimensional representation from Example 1.4.1, and the two dimensional representation from Example 1.4.2 are the only irreducible representations of S_3. Note that S_3 has three conjugacy classes: $\{e\}$, $\{(12), (23), (31)\}$, and $\{(123), (321)\}$. With this, we can construct the *character table* for S_3, with each row containing the characters of the group elements with respect to the given representation.

S_3	e	(12)	(23)	(31)	(123)	(321)
Trivial	1	1	1	1	1	1
Sign	1	−1	−1	−1	1	1
Standard	2	0	0	0	−1	−1

Observe that the rows of this table are orthogonal; indeed, so are the columns!

4.3. The character table for S_4

Let S_4 act on $\mathbb{C}^4$ by permuting the basis vectors $\{e_1, e_2, e_3, e_4\}$, and let (σ, V) denote the induced (natural) representation. Note that each matrix $\sigma(g)$ is a permutation, hence its trace $\chi_V(g)$ is precisely the number of elements of $\{1, 2, 3, 4\}$ fixed by the action of g. With this, we can compute χ_V for each conjugacy class (identified by its cycle type) as follows.

S_4	e	$(ab) \times 6$	$(ab)(cd) \times 3$	$(abc) \times 8$	$(abcd) \times 6$
Trivial	1	1	1	1	1
Sign	1	−1	1	1	−1
V	4	2	0	1	0

Compute

$$\langle \chi_V, \chi_V \rangle = \frac{1}{24}(4^2 + 6 \cdot 2^2 + 8 \cdot 1^2) = 2,$$

whence V is not irreducible. Indeed, we know that $W_1 = \operatorname{span}\{e_1 + e_2 + e_3 + e_4\}$ is a trivial subrepresentation of V of dimension 1. Furthermore, $2 = 1^2 + 1^2$ is the only way of writing 2 as a sum of squares of integers, so V must decompose into precisely two irreducible subrepresentations with both multiplicities 1. This means that $V \cong W_1 \oplus W_3$ for some irreducible representation W_3 of dimension 3. Using $\chi_V = \chi_{W_1} + \chi_{W_3}$, we can compute the character χ_{W_3} and obtain the following.

S_4	e	$(ab) \times 6$	$(ab)(cd) \times 3$	$(abc) \times 8$	$(abcd) \times 6$
W_3	3	1	−1	0	−1

Next, we move on to a different representation of S_4: consider all subsets of size 2 of $\{1, 2, 3, 4\}$ (of which there are 6), and consider the action on this collection induced by the permutations on the set $\{1, 2, 3, 4\}$.

Remark: If we wish to define a transitive action of G on a set X (and thereby examine the vector space $\operatorname{span}\{e_x\}_{x \in X}$ with the action of G defined via $ge_x = e_{gx}$), we may invoke the Orbit-Stabilizer Theorem, along with the fact that there is only one orbit (all of X) to demand that $\operatorname{ord}(X) \mid \operatorname{ord}(G)$.

Let (τ, V') denote the induced representation. Again, $\chi_{V'}(g)$ is the number of 2-subsets fixed by the action of g. For instance, an element $(ab) \in S_4$ will only fix 2-subsets $\{a, b\}, \{c, d\}$, while an element $(abc) \in S_4$ fixes no 2-subset. With this, we have the following.

S_4	e	$(ab) \times 6$	$(ab)(cd) \times 3$	$(abc) \times 8$	$(abcd) \times 6$
V'	6	2	2	0	0

Compute $\langle \chi_{V'}, \chi_{V'} \rangle = 3 = 1^2 + 1^2 + 1^2$. Again, we may compute $\langle \chi_{V'}, \chi_{W_1} \rangle = 1$ and $\langle \chi_{V'}, \chi_{W_3} \rangle = 1$, which tells us that $V' \cong W_1 \oplus W_3 \oplus W_2$ for some irreducible representation W_2 of dimension 2. Using $\chi_{V'} = \chi_{W_1} + \chi_{W_3} + \chi_{W_2}$, we can compute the character χ_{W_2}.

S_4	e	$(ab) \times 6$	$(ab)(cd) \times 3$	$(abc) \times 8$	$(abcd) \times 6$
W_2	2	0	2	−1	0

We now have 4 irreducible characters of S_4; indeed, we may combine W_3 with the sign representation to get another irreducible representation W_3', completing the character table of S_4.

S_4	e	$(ab) \times 6$	$(ab)(cd) \times 3$	$(abc) \times 8$	$(abcd) \times 6$
Trivial	1	1	1	1	1
Sign	1	−1	1	1	−1
W_2	2	0	2	−1	0
W_3	3	1	−1	0	−1
W_3'	3	−1	−1	0	1

The last trick uses the following proposition.

Proposition 4.7: Let (σ, V) and $(\tau, \mathbb{C}^\times)$ be representations of G. Then, $(\tau\sigma, V)$ is also a representation of G, where $(\tau\sigma)(g) = \tau(g)\sigma(g)$. Furthermore, $\chi_{\tau\sigma} = \chi_\tau \chi_\sigma$.

Remark: The above proposition is a special case of Proposition 6.2.

5. Regular representations revisited

We focus our attention once again to the regular representation, as defined in Example 1.1.2. Note that when G acts on itself by left multiplication, only the identity element 1 fixes all $\operatorname{ord}(G)$ elements of G, while the remaining elements have no fixed points at all. With this, we have the following proposition.

Proposition 5.1: Let (τ, V_G) be the regular representation of G, and let χ_τ denote its character. Then,

$$\chi_\tau(g) = \begin{cases} \operatorname{ord}(G) & \text{if } g = 1 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\chi_\tau = d_1\chi_1 + \ldots + d_k\chi_k$$

where $\chi_1, \ldots, \chi_k$ are the irreducible characters of G, and each $d_i = \chi_i(1)$ is the dimension of the corresponding irreducible representation.

By simply evaluating $\chi_\tau(1)$, we have the following result.

Corollary 5.1.1: Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of G, and let each $d_i = \chi_i(1)$. Then,

$$\sum_i d_i^2 = \operatorname{ord}(G).$$

Proposition 5.2: Let $f : G \to \mathbb{C}$ be a class function, and let (σ, V) be a representation of G. Define

$$f_\sigma : V \to V, \qquad v \mapsto \sum_{g \in G} f(g)\, \sigma(g)(v).$$

Then, f_σ is G-invariant. Furthermore, if (σ, V) is irreducible, then Schur's Lemma (Theorem 3.3) gives $f_\sigma = \lambda\, \mathrm{id}_V$, where $\lambda = \operatorname{ord}(\mathrm{G})\ \langle f, \overline{\chi_\sigma} \rangle / \dim(V)$.

With this construction, we can improve upon Corollary 4.5.1 and precisely count the number of irreducible representations of a group G (up to isomorphism). However, we still do not have any simple way of calculating these representations explicitly.

Theorem 5.3: The number of irreducible representations of G (up to isomorphism) is precisely the number of conjugacy classes of G.

Proof: Let $\mathcal{C}$ be the space of class functions on G, with dimension equal to the number of conjugacy classes of G. Let $\mathcal{X}$ be the subspace of $\mathcal{C}$ spanned by the irreducible characters $\{\chi_i\}$ of G. We claim that $\mathcal{X} = \mathcal{C}$. It is enough to show that the orthocomplement of $\mathcal{X}$ in $\mathcal{C}$ is trivial. For this, pick $f \in \mathcal{C}$ such that all $\langle f, \overline{\chi_i} \rangle = 0$. Let (τ, V_G) be the regular representation of G, and use Proposition 5.1 to write

$$V_G \cong d_1 V_1 \oplus \ldots \oplus d_k V_k$$

where $\{(\sigma_i, V_i)\}$ are the irreducible representations corresponding to the characters $\{\chi_i\}$. Using Proposition 5.2, each $f_{\sigma_i} = 0$, hence $f_\tau = 0$. Evaluating f_τ at the element $e_1 \in V_G$, we have

$$\sum_{g \in G} f(g)\, \sigma(g)(e_1) = \sum_{g \in G} f(g)\, e_g = 0.$$

Since $\{e_g\}_{g\in G}$ forms a basis of V_G, we must have $f = 0$. □

Corollary 5.3.1: The irreducible characters of G form a basis of the space of all class functions on G.

6. Tensor products of representations

The construction used in Proposition 4.7 generalizes nicely to tensor products of representations, as follows.

Definition 6.1: Let V, V' be two representations of G. Then, the tensor product $V \otimes V'$ is a representation of G induced by the action defined by

$$g(v \otimes v') = (gv) \otimes (gv').$$

Recall that if $\{v_i\}$ is a basis of V and $\{v'_j\}$ is a basis of V', then $\{v_i \otimes v'_j\}$ forms a basis of $V \otimes V'$. Using this, the next proposition follows.

Proposition 6.2: Let V, V' be two representations of G. Then, $\chi_{V\otimes V'} = \chi_V \chi_{V'}$.

Let's focus on the tensor product $V \otimes V$ and examine the involution defined by

$$\iota : V \otimes V \to V \otimes V, \qquad v \otimes v' \mapsto v' \otimes v.$$

This map has two eigenspaces, corresponding to the eigenvalues 1 and -1, which we define as $\mathrm{Sym}^2(V)$ and $\mathrm{Alt}^2(V)$ respectively. Furthermore, it is clear that these eigenspaces are stable under the action of G, since the action of G commutes with ι. This gives us the following decomposition of $V \otimes V$.

Proposition 6.3: Let V be a representation of G. Then,

$$V \otimes V \cong \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V).$$

Observe that $\mathrm{Sym}^2(V)$ is spanned by elements of the form $v \otimes v' + v' \otimes v$, while $\mathrm{Alt}^2(V)$ is spanned by elements of the form $v \otimes v' - v' \otimes v$. This, together with $\dim(V \otimes V) = n^2$, tells us that

$$\dim(\mathrm{Sym}^2(V)) = \binom{n}{2} + n, \qquad \dim(\mathrm{Alt}^2(V)) = \binom{n}{2}.$$

To compute the characters of these representations, first note that

$$\chi_V^2 = \chi_{\mathrm{Sym}^2(V)} + \chi_{\mathrm{Alt}^2(V)}.$$

Fix $g \in G$ and choose a basis $\{v_i\}$ of V such that the action of g is diagonalized, i.e. $gv_i = \lambda_i v_i$; recall that this is always possible via Proposition 1.5 when we are working over the field $K = \mathbb{C}$. We need only check the action of g on the basis elements of $\mathrm{Sym}^2(V)$ and $\mathrm{Alt}^2(V)$. To this end, compute

$$g(v_i \otimes v_j + v_j \otimes v_i) = \lambda_i \lambda_j (v_i \otimes v_j + v_j \otimes v_i),$$

whence

$$\chi_{\mathrm{Sym}^2(V)}(g) = \sum_{i \le j} \lambda_i \lambda_j = \sum_{i<j} \lambda_i \lambda_j + \sum_i \lambda_i^2 = \frac{1}{2}\left(\sum_i \lambda_i\right)^2 + \frac{1}{2}\sum_i \lambda_i^2.$$

However, $\sum_i \lambda_i$ and $\sum_i \lambda_i^2$ are precisely $\chi_V(g)$ and $\chi_V(g^2)$. Thus, we have the following result.

Proposition 6.4:

$$\chi_{\mathrm{Sym}^2(V)}(g) = \frac{1}{2}\left(\chi_V(g)^2 + \chi_V(g^2)\right), \qquad \chi_{\mathrm{Alt}^2(V)}(g) = \frac{1}{2}\left(\chi_V(g)^2 - \chi_V(g^2)\right).$$

6.1. The character table for S_5

We proceed much in the same manner as in the computation for S_4; the character of the natural representation V of S_5 (induced by the action of S_5 on 5 objects) can be computed by counting the number of fixed points of the corresponding conjugacy class.

S_5	e	(ab)	$(ab)(cd)$	(abc)	$(ab)(cdf)$	$(abcd)$	$(abcdf)$
	$\times 1$	$\times 10$	$\times 15$	$\times 20$	$\times 20$	$\times 30$	$\times 24$
Trivial	1	1	1	1	1	1	1
Sign	1	−1	1	1	−1	−1	1
V	5	3	1	2	0	1	0

Since the inner product of the first and last rows is non-zero, V contains a copy of the trivial representation, which we subtract from its character to obtain the representation W_4. This happens to be irreducible (verified by checking $\langle \chi_W, \chi_W \rangle = 1$), and we obtain another irreducible representation W_4' by taking the tensor product with the sign representation.

S_5	e	(ab)	$(ab)(cd)$	(abc)	$(ab)(cdf)$	$(abcd)$	$(abcdf)$
W_4	4	2	0	1	−1	0	−1
W_4'	4	−2	0	1	1	0	−1

We now compute the characters of $\mathrm{Sym}^2(W_4)$ and $\mathrm{Alt}^2(W_4)$ using Proposition 6.4. To do so, we compute $\chi_{W_4}(g)^2$ and $\chi_{W_4}(g^2)$ for each conjugacy class; the latter can be computed by examining what happens to each cycle type after squaring.

S_5	e	(ab)	$(ab)(cd)$	(abc)	$(ab)(cdf)$	$(abcd)$	$(abcdf)$
$\chi_{W_4}^2$	16	4	0	1	1	0	1
$\chi_{W_4}(g^2)$	4	4	4	1	1	0	−1
$\mathrm{Alt}^2(W_4)$	6	0	−2	0	0	0	1
$\mathrm{Sym}^2(W_4)$	10	4	2	1	1	0	0

Note that $\mathrm{Alt}^2(W_4)$ is irreducible, but $\mathrm{Sym}^2(W_4)$ is not. Indeed, the latter clearly has a positive inner product with the trivial representation, hence contains a copy of it which we take away giving us W_9.

S_5	e	(ab)	$(ab)(cd)$	(abc)	$(ab)(cdf)$	$(abcd)$	$(abcdf)$
W_9	9	3	1	0	0	−1	−1

Looking back, we have found five irreducible representations of S_5, and hence are left to discover two more. The sum of squares of their dimensions is 70; Corollary 5.1.1 tells us that the sum of squares of the dimensions of the remaining two must be $\mathrm{ord}(S_5) - 70 = 50$. Now, $50 = 1^2 + 7^2 = 5^2 + 5^2$, but the first decomposition is not possible since the trivial and sign representations are the only ones of dimension 1. Thus, both remaining representations are of dimension 5.

Observe that $\langle \chi_{W_9}, \chi_{W_9} \rangle = 2 = 1^2 + 1^2$, hence W_9 must be composed of two irreducible representations. The only way to write 9 as the sum of two dimensions of irreducible representations (which we know are $1, 1, 4, 4, 5, 5, 6$) is $4 + 5$. Indeed, $\langle \chi_{W_9}, \chi_{W_4} \rangle = 1$, so we take W_4 away leaving us with an irreducible representation W_5. Taking the tensor product with the sign representation yields our final irreducible representation $W_{5'}$. Thus, the complete character table of S_5 is as follows.

S_5	e	(ab)	$(ab)(cd)$	(abc)	$(ab)(cdf)$	$(abcd)$	$(abcdf)$
	$\times$ 1	$\times$ 10	$\times$ 15	$\times$ 20	$\times$ 20	$\times$ 30	$\times$ 24
Trivial	1	1	1	1	1	1	1
Sign	1	−1	1	1	−1	−1	1
W_4	4	2	0	1	−1	0	−1
W_4'	4	−2	0	1	1	0	−1
W_5	5	1	1	−1	1	−1	0
W_5'	5	−1	1	−1	−1	1	0
$\mathrm{Alt}^2(W_4)$	6	0	−2	0	0	0	1

Remark: Observe that the columns of this character table are pairwise orthogonal! Indeed, we already know from Proposition 5.1 that the first column of any character table must be orthogonal to the rest. Furthermore, this character table strongly suggests a relation of the form

$$\sum_i |\chi_i(g)|^2 = \frac{\mathrm{ord}(G)}{\mathrm{ord}(\mathrm{Cl}(g))},$$

where $\chi_i, \ldots, \chi_k$ are the irreducible characters of G, and $\mathrm{Cl}(g)$ denotes the conjugacy class of $g \in G$.

7. Orthogonality of characters revisited

Let $\mathcal{O}_1, \ldots, \mathcal{O}_k$ be the conjugacy classes of G, let $n_1, \ldots, n_k$ be their sizes, and let $\chi_1, \ldots, \chi_k$ be the irreducible characters of G. Furthermore, let's work with the field $K = \mathbb{C}$. Set $n = \mathrm{ord}(G)$. We know from Theorem 4.5 that (with a little abuse of notation)

$$\sum_\ell n_\ell \, \chi_i(\mathcal{O}_\ell)\overline{\chi_j(\mathcal{O}_\ell)} = \delta_{ij} n.$$

We construct the matrix $A \in M_k(\mathbb{C})$ via $A_{i\ell} = \sqrt{n_\ell / n}\, \chi_i(\mathcal{O}_\ell)$ and note that $AA^\dagger = \mathbb{I}_n$. Thus, A is a unitary matrix with orthonormal rows. As a result, the columns of A must also be orthonormal, giving

$$\delta_{ij} = \sum_\ell A_{\ell i} \overline{A_{\ell j}} = \frac{\sqrt{n_i n_j}}{n} \sum_\ell \chi_\ell(\mathcal{O}_i) \overline{\chi_\ell(\mathcal{O}_j)}.$$

In other words,

$$\sum_\ell \chi_\ell(\mathcal{O}_i) \overline{\chi_\ell(\mathcal{O}_j)} = \begin{cases} n/n_i & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the columns of the character table of G also obey an orthonormality relation. We supply a different proof below.

Theorem 7.1 (Column orthogonality of characters): Suppose that $K = \mathbb{C}$. Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of G, and let $g, g' \in G$. Denote $g \sim g'$ if g and g' are conjugates. Then,

$$\sum_i \chi_i(g) \overline{\chi_i(g')} = \begin{cases} \operatorname{ord}(G) / \operatorname{ord}(\operatorname{Cl}(g)) & \text{if } g \sim g' \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Define the class function

$$f_g : G \to \mathbb{C}, \qquad h \mapsto \begin{cases} 1 & \text{if } g \sim h \\ 0 & \text{otherwise.} \end{cases}$$

Using Corollary 5.3.1, we can expand f_g in terms of the irreducible characters as

$$f_g = a_1 \chi_1 + \ldots + a_k \chi_k,$$

and compute each $a_i = \langle f_g, \chi_i \rangle = \operatorname{ord}(\operatorname{Cl}(g)) \overline{\chi_i(g)} / \operatorname{ord}(G)$. Thus,

$$\frac{\operatorname{ord}(\operatorname{Cl}(g))}{\operatorname{ord}(G)} \sum_i \chi_i(g) \overline{\chi_i(h)} = \overline{f_g(h)} = \begin{cases} 1 & \text{if } g \sim h \\ 0 & \text{otherwise} \end{cases}$$ □

7.1. The character table for A_5

Note that A_5 has five conjugacy classes; the 5-cycles split into two conjugacy classes in A_5. As usual, we start with the natural representation V of A_5, then take away the copy of the trivial representation yielding an irreducible representation W_4.

A_5	e	$(12)(34)$	(123)	(12345)	(13524)
	$\times 1$	$\times 15$	$\times 20$	$\times 12$	$\times 12$
Trivial	1	1	1	1	1
V	5	1	2	0	0
W_4	4	0	1	−1	−1

We compute $\operatorname{Sym}^2(W_4)$ and $\operatorname{Alt}^2(W_4)$ as usual.

A_5	e	$(12)(34)$	(123)	(12345)	(13524)
$\chi^2_{W_4}$	16	0	1	1	1
$\chi_{W_4}(g^2)$	4	4	1	−1	−1
$\mathrm{Alt}^2(W_4)$	6	−2	0	1	1
$\mathrm{Sym}^2(W_4)$	10	2	1	0	0

Now, $\mathrm{Sym}^2(W_4)$ contains copies of both the trivial representation and W_4; taking them away yields the irreducible representation W_5.

A_5	e	$(12)(34)$	(123)	(12345)	(13524)
W_5	5	1	−1	0	0

Looking back, we have found three out of five irreducible representations. The sum of squares of their dimensions is 42, hence the sum of squares dimensions of the remaining two representations must be $\mathrm{ord}(A_5) - 42 = 18$ by Corollary 5.1.1. The only way to write this as a sum of two squares is $18 = 3^2 + 3^2$, hence the remaining two irreducible representations must both have dimension 3.

We can now fill in most of the character table for A_5 as follows.

A_5	e	$(12)(34)$	(123)	(12345)	(13524)
	$\times 1$	$\times 15$	$\times 20$	$\times 12$	$\times 12$
Trivial	1	1	1	1	1
W_4	4	0	1	−1	−1
W_3	3	a	b	c	d
W_3'	3	a'	b'	c'	d'
W_5	5	1	−1	0	0

We can be sure that the unknown quantities are real because of the following observation.

Lemma 7.2: Let G be such that for every $g \in G$, the elements g and g^{-1} belong to the same conjugacy class. Then, the characters of G (with $K = \mathbb{C}$) are real valued.

Proof: Use $\chi(g) = \chi(g^{-1}) = \overline{\chi(g)}$, with the first equality following from the fact that χ is a class function and $g \sim g^{-1}$. □

Column orthogonality (Theorem 7.1) gives the relations $1 + 3a + 3a' + 5 = 0$ and $1^2 + a^2 + {a'}^2 + 1^2 = 60/15 = 4$, which simplifies to $a + a' = -2$ and $a^2 + {a'}^2 = 2$. This is only possible if $a = a' = -1$. Again, we have the relations $1 + 4 + 3b + 3b' - 5 = 0$ and $1^2 + 1^2 + b^2 + {b'}^2 + 1^2 = 60/20 = 3$, which simplify to $b + b' = 0$ and $b^2 + {b'}^2 = 0$. This is only possible if $b = b' = 0$.

Similar computations give $c + c' = d + d' = 1$, and $c^2 + {c'}^2 = d^2 + {d'}^2 = 3$. This gives $cc' = dd' = -1$. Thus, c, c' and d, d' are the roots of $x^2 - x - 1$, which we denote as $\varphi = \left(1 + \sqrt{5}\right)/2$ and $\psi = \left(1 - \sqrt{5}\right)/2$. Without loss of generality, set $c = \varphi$, $c' = \psi$. Using the column orthogonality $1 + 1 +$

$cd + c'd' = 0$, we discard the case $d = \varphi$, $d' = \psi$, leaving $d = \psi$, $d' = \varphi$. This completes our computation of the character table of A_5.

A_5	e	$(12)(34)$	(123)	(12345)	(13524)
	$\times 1$	$\times 15$	$\times 20$	$\times 12$	$\times 12$
Trivial	1	1	1	1	1
W_4	4	0	1	−1	−1
W_3	3	−1	0	φ	ψ
W_3'	3	−1	0	ψ	φ
W_5	5	1	−1	0	0

8. Induced representations

Consider a representation W of a subgroup $H \subseteq G$. We want to obtain a representation V of G, by extending the action of H on W to the action of G on a larger space. To do so, we first let $\mathcal{R} = \{r_1, ..., r_m\}$ be representative elements from each of the cosets of H in G, and consider the vector spaces $r_i W$. These are just 'copies' of W, containing elements of the form $r_i w$ for $w \in W$. We have the isomorphisms

$$\varphi_i : W \to r_i W_i, \qquad w \mapsto r_i w.$$

With this, define the vector space

$$V = r_1 W \oplus ... \oplus r_m W.$$

We will define the action of G on V via the components $r_i W$. Given $g \in G$, write $g r_i = r h$ for $r \in \mathcal{R}$, $h \in H$ (this is always possible since g belongs to precisely one coset of H in G). With this, for $r_i w \in r_i W$, define $g(r_i w) = r(hw)$. It can be shown that this does indeed define a group action of G on V. We call the associated representation V of G the induced representation $\mathrm{Ind}_H^G(W)$.

For some fixed $g \in G$, observe that the action of g permutes the cosets $r_i W$ among each other. In particular, if $g r_i = r h$, we have $g(r_i W) = r W$. Thus, if we represent g as a block matrix using the natural basis $\{r_i w_j\}$ (where $\{w_j\}$ forms a basis of W), the result contains blocks of size $\dim(W)$, with exactly one block per 'row' and 'column'. This means that when computing the trace of g, the only blocks that contribute correspond to those i where $g(r_i W) = r_i W$, i.e. $g r_i = r_i h$ for some $h \in H$. In this setting, we have each $g(r_i w) = r_i(hw)$; thus, the trace of g acting on $r_i W$ is precisely the same as the trace of h acting on W (use the isomorphism φ_i to note that $g = \varphi_i \circ h \circ \varphi_i^{-1}$ on $r_i W$). Since $h = r_i^{-1} g r_i$, we have

$$\chi_V(g) = \sum_{\substack{r \in \mathcal{R} \\ r^{-1} g r \in H}} \chi_W(r^{-1} g r) = \frac{1}{\mathrm{ord}(H)} \sum_{\substack{s \in G \\ s^{-1} g s \in H}} \chi_W(s^{-1} g s).$$

The last equivalence follows from the fact that if $s \in rH$, then we may write $s = rh$ for some $h \in H$, hence $s^{-1} g s = h^{-1}(r^{-1} g r) h \in H$ if and only if $r^{-1} g r \in H$; furthermore, χ_W will be the same for both of them since they are conjugates in H. The factor of $\mathrm{ord}(H) = \mathrm{ord}(rH)$ accounts for the number of these repeated terms.

Definition 8.1: Let W be a representation of a subgroup H of G. The induced character $\operatorname{Ind}_H^G(\chi_W)$ of χ_W is defined as

$$\operatorname{Ind}_H^G(\chi_W)(g) = \frac{1}{\operatorname{ord}(H)} \sum_{\substack{s\in G \\ s^{-1}gs\in H}} \chi_W(s^{-1}gs).$$

Observe that as a map,

$$\operatorname{Ind}_H^G : \mathcal{C}_H \to \mathcal{C}_G$$

is maps class functions on H to class functions on G.

Suppose that $\mathcal{O}$ is a conjugacy class in H. We may define the class function

$$\chi_{\mathcal{O}} : H \to \mathbb{C}, \qquad g \mapsto \mathbf{1}_{\mathcal{O}}(g) = \begin{cases} 1 & \text{if } g \in \mathcal{O} \\ 0 & \text{otherwise} \end{cases}.$$

Note that there is a unique conjugacy class $\tilde{\mathcal{O}}$ in G containing $\mathcal{O}$. Now, $\operatorname{Ind}_H^G(\chi_{\mathcal{O}})$ must vanish outside $\tilde{\mathcal{O}}$; since it is constant on $\tilde{\mathcal{O}}$, we must have $\operatorname{Ind}_H^G(\chi_{\mathcal{O}}) = \lambda\chi_{\tilde{\mathcal{O}}}$ for some scalar λ. We compute λ by evaluating at some $g \in \mathcal{O}$, whence

$$\begin{aligned}
\lambda = \operatorname{Ind}_H^G(\chi_{\mathcal{O}}) &= \frac{1}{\operatorname{ord}(H)} \sum_{\substack{s\in G \\ s^{-1}gs\in H}} \chi_{\mathcal{O}}(s^{-1}gs) \\
&= \frac{\operatorname{ord}(\{s \in G : s^{-1}gs \in \mathcal{O}\})}{\operatorname{ord}(H)} \\
&= \frac{\operatorname{ord}(\mathcal{O}) \cdot \operatorname{ord}(Z_G(g))}{\operatorname{ord}(H)} \\
&= \frac{\operatorname{ord}(G)}{\operatorname{ord}(H)} \cdot \frac{\operatorname{ord}(\mathcal{O})}{\operatorname{ord}\left(\tilde{\mathcal{O}}\right)}.
\end{aligned}$$

Here, $Z_G(g)$ denotes the centralizer of g in G. Together with the orbit-stabilizer theorem, This gives us the following.

Lemma 8.2: Let $\mathcal{O}$ be a conjugacy class in $H \subseteq G$, let $\tilde{\mathcal{O}}$ be the unique conjugacy class in G containing $\mathcal{O}$, and let $g \in \mathcal{O}$. Then,

$$\operatorname{Ind}_H^G(\chi_{\mathcal{O}}) = \frac{\operatorname{ord}(Z_G(g))}{\operatorname{ord}(Z_H(g))}\chi_{\tilde{\mathcal{O}}}.$$

Theorem 8.3 (Frobenius reciprocity): Let χ be a class function on $H \subseteq G$, let χ' be a class function on G, and let $\tilde{\chi}$ denote the restriction of χ' to H. Then,

$$\langle \operatorname{Ind}_H^G(\chi), \chi' \rangle_G = \langle \chi, \tilde{\chi} \rangle_H.$$

Proof: It is sufficient to prove this for all χ of the form $\chi_{\mathcal{O}}$ for conjugacy classes $\mathcal{O}$ in H, since they form a basis of $\mathcal{C}_H$. The result is immediate from $\langle \chi_{\mathcal{O}}, \tilde{\chi} \rangle_H = \overline{\chi'(\mathcal{O})} \cdot \operatorname{ord}(\mathcal{O}) / \operatorname{ord}(H)$ and Lemma 8.2. □

8.1. The character table for $\mathrm{GL}_2(\mathbb{F}_q)$

Let $\mathbb{F}_q$ denote the finite field of order $q = p^n$, where p is prime. Recall that $\mathbb{F}_q^\times = \mathbb{F}_q \setminus \{0\}$ is a cyclic group of order $q - 1$.

Proposition 8.4: Let $[K : \mathbb{F}_q] = n$. Then, $K \cong \mathbb{F}_{q^n}$.

Henceforth, we are interested in the group $G = \mathrm{GL}_2(\mathbb{F}_q)$.

Lemma 8.5: The group $\mathrm{GL}_2(\mathbb{F}_q)$ has size $(q^2 - 1)(q^2 - q) = q(q + 1)(q - 1)^2$.

Proof: Each of the $q^2 - 1$ non-zero choices for the first column of an element of $\mathrm{GL}_2(\mathbb{F}_q)$ has q multiples, leaving $q^2 - q$ choices for the second column. □

We compute and classify the conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)$ as follows. Fixing $\varepsilon \in \mathbb{F}_q \setminus \{x^2 : x \in \mathbb{F}_q\}$, we have the following.

	Matrix type	Size of centralizer	Size of conjugacy class	Number of classes
I	$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ $\lambda \in \mathbb{F}_q^\times$	$\operatorname{ord}(G)$	1	$q - 1$
II	$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ $\lambda \in \mathbb{F}_q^\times$	$q(q-1)$	$q^2 - 1$	$q - 1$
III	$\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ $\lambda, \mu \in \mathbb{F}_q^\times$ $\lambda \neq \mu$	$(q-1)^2$	$q(q+1)$	$\frac{(q-1)(q-2)}{2}$
IV	$\begin{bmatrix} a & b\varepsilon \\ b & a \end{bmatrix}$ $a, b \in \mathbb{F}_q$ $b \neq 0$	$q^2 - 1$	$q(q-1)$	$\frac{q(q-1)}{2}$

These can be deduced by considering the possible minimal polynomials f for an element M of $\mathrm{GL}_2(\mathbb{F}_q)$. If $f(x) = x - \lambda$, then clearly $M = \lambda \mathbb{I}_2$, hence we have **Type I**.

Suppose that $f(x) = (x - \lambda)^2$. It follows by evaluating $f(0), f(1)$ that $2\lambda, \lambda^2 \in \mathbb{F}_q$, hence $\lambda \in \mathbb{F}_q$.

Lemma 8.6: Every matrix with minimal polynomial $(x-\lambda)^2$ is conjugate to $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

Proof: Let M be such a matrix, hence $N^2 = 0$ where $N = M - \lambda$. Furthermore, $\operatorname{im}(N) = \ker(N)$, with $\dim(\operatorname{im}(N)) = 1$. Let $\operatorname{im}(N) = \operatorname{span}\{v\}$, and pick $w \notin \operatorname{im}(N)$. Then, $Nv = 0$, and $Nw \in \operatorname{im}(V)$, hence $Nw = \mu v$ for some $\mu \in \mathbb{F}_q$. Thus, $N \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ via the basis $\{v, w/\mu\}$. With this, write $N = P\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}P^{-1}$, so $M = N + \lambda = P\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}P^{-1}$. □

This gives us **Type II**. The centralizer of M is computed as follows: A commutes with M if and only if it commutes with $N = M - \lambda$. Setting $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we demand $c = 0$ and $a = d$, which can be fulfilled in $q(q-1)$ ways (non-trivially).

If $f(x) = (x-\lambda)(x-\mu)$ for distinct $\lambda, \mu \in \mathbb{F}_q$, we must have $M \sim \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$, giving us **Type III**. Computing the centralizer by brute force, we see that it consists of diagonal matrices.

Finally, if f is irreducible over $\mathbb{F}_q$, note that f must split over $\mathbb{F}_{q^2}$, the unique degree 2 extension of $\mathbb{F}_q$. Our choice of ε ensures that $\mathbb{F}_{q^2} = \mathbb{F}_q(\sqrt{\varepsilon})$. Thus, we may write $f(x) = (x-\alpha)(x-\beta)$ where $\alpha, \beta \in \mathbb{F}_q(\sqrt{\varepsilon})$ are conjugate. Let $\alpha = a + b\sqrt{\varepsilon}$, $\beta = a - b\sqrt{\varepsilon}$ for $a, b \in \mathbb{F}_q$, $b \neq 0$. Now, $M \sim \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ in $\mathrm{GL}_2(\mathbb{F}_{q^2})$. Find $v \in \mathbb{F}_{q^2}^2$ such that $Mv = \alpha v$; applying Galois conjugation, we have $M\overline{v} = \overline{\alpha v} = \beta\overline{v}$, hence $Mw = \beta w$ for $w = \overline{v}$. With this, $v + w, \sqrt{\varepsilon}(v - w) \in \mathbb{F}_q^2$ are linearly independent! We may compute that in this basis, $M = \begin{bmatrix} a & b\varepsilon \\ b & a \end{bmatrix}$, giving us **Type IV**.

Proposition 8.7: The centralizer of $\begin{bmatrix} a & b\varepsilon \\ b & a \end{bmatrix}$ contains all matrices $\begin{bmatrix} c & d\varepsilon \\ d & c \end{bmatrix}$.

Thus, the centralizer of M has size at least $q^2 - 1$, whence there are at most $\operatorname{ord}(G)/(q^2-1) = q(q-1)$ members of the same conjugacy class. The class equation will force this to be an equality. Indeed, the previous proposition identifies *all* elements of the centralizer of M.

9. The group ring

Definition 9.1 (Group ring): Let G be a group. Its group ring $\mathbb{C}[G]$ is described as the $\mathbb{C}$-span of the elements of G, with the multiplicative structure inherited from G.

In particular, every element of $\mathbb{C}[G]$ is of the form $\sum_{g \in G} \lambda_g g$. The product of two such elements can be written as

$$\left(\sum_{g \in G} \lambda_g g\right)\left(\sum_{g' \in G} \mu_{g'} g'\right) = \sum_{g, g' \in G} \lambda_g \mu_{g'} g g'.$$

The elements

$$e_{\mathcal{O}} = \sum_{g \in \mathcal{O}} g$$

for conjugacy classes $\mathcal{O}$ in G are of particular interest.

Lemma 9.2: The elements $\{e_{\mathcal{O}}\}$ form a basis of the center of $\mathbb{C}[G]$.

Proof: Observe that $s = \sum_{g \in G} \lambda_g g$ is in the center of $\mathbb{C}[G]$ if and only if for all $g_0 \in G$, we have

$$g_0 \left(\sum_{g \in G} \lambda_g g \right) = \left(\sum_{g \in G} \lambda_g g \right) g_0,$$

which reduces to $\lambda_g = \lambda_{g_0^{-1} g g_0}$ for all $g, g_0 \in G$. In other words, $\lambda_g = \lambda_{g'}$ if and only if $g \sim g'$. However, this is precisely the condition for $s \in \operatorname{span}_{\mathbb{C}}\{e_{\mathcal{O}}\}$. □

It is clear that representations of G, i.e. group homomorphisms $\sigma : G \to \operatorname{GL}(V)$ extend naturally to ring homomorphisms

$$\tilde{\sigma} : \mathbb{C}[G] \to \mathcal{L}(V), \qquad \sum_{g \in G} \lambda_g g \mapsto \sum_{g \in G} \lambda_g \sigma(g).$$

Suppose that $s = \sum_{g \in G} \lambda_g g$ is in the center of $\mathbb{C}[G]$. It follows that $\tilde{\sigma}(s)$ commutes with $\tilde{\sigma}(g) = \sigma(g)$ for all $g \in G$. Thus, $\tilde{\sigma}(s) \in \mathcal{L}(V)$ is a G-invariant map. Furthermore, when V is irreducible, Schur's Lemma (Theorem 3.3) tells us that $\tilde{\sigma}(s)$ is a scalar mad of the form $\lambda \operatorname{id}_V$. To compute λ, we take the trace of $\tilde{\sigma}(s) = \sum_{g \in G} \lambda_g \sigma(g)$ yielding

$$\sum_{g \in G} \lambda_g \chi_V(g) = \lambda \dim(V).$$

Lemma 9.3: Let V be an irreducible representation of G, and let $s = \sum_{\mathcal{O}} \lambda_{\mathcal{O}} e_{\mathcal{O}}$. Then,

$$\tilde{\sigma}(s) = \frac{1}{\dim(V)} \sum_{\mathcal{O}} \lambda_{\mathcal{O}} \operatorname{ord}(\mathcal{O}) \chi_V(\mathcal{O}) \cdot \operatorname{id}_V .$$

9.1. Integral elements

We now take a brief detour and recall a few facts about integral ring extensions.

Definition 9.4 (Integral extension): Let R be a commutative ring containing $\mathbb{Z}$. An element x is called integral over $\mathbb{Z}$ if it satisfies

$$x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

for some $a_i \in \mathbb{Z}$.

Example 9.4.1: Let $R = \mathbb{Q}$. Then, the integral elements over $\mathbb{Z}$ in $\mathbb{Q}$ are precisely the integers $\mathbb{Z}$. To see this, note that if $x = p/q$ for coprime p, q satisfies

$$x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0,$$

then we have

$$p^n + [a_{n-1} p^{n-1} q + \dots + a_1 p q^{n-1} + a_0 q^n] = 0$$

where the bracketed term is divisible by q, but the leading term is not unless $q = 1$.

Definition 9.5 (Algebraic integers): Integral elements in $\mathbb{C}$ over $\mathbb{Z}$ are called algebraic integers.

Lemma 9.6: The following are equivalent.

1. $x \in R$ is integral over $\mathbb{Z}$.
2. $\mathbb{Z}[x] \subset R$ is finitely generated as a Z-module.

Proof: To show ($\Longrightarrow$), write $x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0$; we now claim that $\mathbb{Z}[x]$ is spanned by $\{1, x, \ldots, x^{n-1}\}$. It suffices to check that every x^m can be written as a linear combination of $\{1, x, \ldots, x^{n-1}\}$. But for $m \geq n$, we may write

$$x^m = -a_{n-1}x^{m-1} - \ldots - a_1 x^{m-n+1} - a_0 x^{m-n},$$

whence the claim follows by induction on m.

To show ($\Longleftarrow$), suppose that $\{f_1, \ldots, f_n\} \subset \mathbb{Z}[x]$ is a finite generating set. Choosing $d > 1$ such that $d > \max_i(\deg(f_i))$, we may expand

$$x^d = a_1 f_1 + \ldots + a_n f_n$$

for $a_i \in \mathbb{Z}$, which gives us the desired condition on x. □

Corollary 9.6.1: The set of integral elements of R over $\mathbb{Z}$ forms a ring.

The proof of the above result uses the fact that if $\mathbb{Z}[x], \mathbb{Z}[y]$ are finitely generated $\mathbb{Z}$-modules, then so are $\mathbb{Z}[x, y]$, $\mathbb{Z}[x + y]$, and $\mathbb{Z}[xy]$.

With this, set

$$R = \left\{ \sum_{\mathcal{O}} n_{\mathcal{O}} e_{\mathcal{O}} : n_{\mathcal{O}} \in \mathbb{Z} \right\} = \bigoplus_{\mathcal{O}} \mathbb{Z} e_{\mathcal{O}}.$$

It is easily checked that R is indeed a ring, sitting inside the center of $\mathbb{C}[G]$. Furthermore, R is a finite dimensional $\mathbb{Z}$-module, whence each $\mathbb{Z}[x]$ for $x \in R$ is a finitely generated $\mathbb{Z}$-module. Thus, each element of R is integral over $\mathbb{Z}$.

In particular, all $e_{\mathcal{O}}$ are integral over $\mathbb{Z}$. Combining this with Corollary 9.6.1 gives us the following.

Proposition 9.7: The element $\sum_{\mathcal{O}} \lambda_{\mathcal{O}} e_{\mathcal{O}}$ is integral over $\mathbb{Z}$ if all $\lambda_{\mathcal{O}}$ are algebraic integers.

Remark: For an algebraic integer λ, note that λe_1 is integral in $\mathbb{C}[G]$ over $\mathbb{Z}$ since it satisfies the same polynomial relation as λ!

Recall the setup of Lemma 9.3: for an irreducible representation (σ, V) of G and an element $s = \sum_{g \in G} \lambda_g g$ in the center of $\mathbb{C}[G]$, we have $\tilde{\sigma}(s) = \lambda_s \operatorname{id}_V$. This gives us a natural map, indeed a homomorphism which by abuse of notation is denoted as

$$\tilde{\sigma} : Z(\mathbb{C}[G]) \to \mathbb{C}, \qquad s \mapsto \lambda_s = \frac{1}{\dim(V)} \sum_{g \in G} \lambda_g \chi_V(g).$$

Here, $Z(\mathbb{C}[G])$ denotes the center of $\mathbb{C}[G]$. Thus, if $s \in Z(\mathbb{C}[G])$ is integral over $\mathbb{Z}$, then λ_s is an algebraic integer!

Theorem 9.8: Let (σ, V) be an irreducible representation of G. Then, $\dim(V) \mid G$.

Proof: Set $s = \sum_{g \in G} \lambda_g g$, and $\lambda_g = \overline{\chi}(g)$. Then, $s \in Z(\mathbb{C}[G])$, whence

$$\tilde{\sigma}(s) = \frac{1}{\dim(V)} \sum_{g \in G} \overline{\chi}(g)\chi(g) = \frac{\operatorname{ord}(G)}{\dim(V)}$$

is an algebraic integer. Furthermore, this is a rational algebraic integer, hence must belong to $\mathbb{Z}$. □

9.2. Burnside's theorem

Theorem 9.9 (Burnside): If $\operatorname{ord}(G) = p^n q^m$ for primes p, q, then G is not simple (unless $\operatorname{ord}(G)$ is prime).

We will prove this theorem using the machinery of group representations. To do so, we first take a closer look at representations of normal subgroups.

Proposition 9.10: Let (σ, V) be a representation of G. Then,

$$\ker(\sigma) = \{g \in G : \chi(g) = \chi(1) = \dim(V)\}$$

is a normal subgroup of G.

Observe in particular that $\chi(g) = \chi(1) = \dim(V)$ forces $\sigma(g) = \mathrm{id}_V$, by applying the triangle inequality on the sum of eigenvalues of $\sigma(g)$.

Proposition 9.11: Let N be a normal subgroup of G, and let σ be a representation of G/N. Then, $\sigma \circ \pi$ is a representation of G, where $\pi : G \to G/N$ is the usual projection map sending elements of G to their respective cosets.

Lemma 9.12: Let α be an algebraic number with minimal polynomial f. Then, α is an algebraic integer if and only if f has integer coefficients.

Proof: ($\Longleftarrow$) Trivial.
($\Longrightarrow$) Let f_α be a monic polynomial with integer coefficients such that $f_\alpha(\alpha) = 0$. Then, we must have $f \mid f_\alpha$, so $f_\alpha = fg$ for another monic polynomial g. Write $\tilde{f} = f/m$, $\tilde{g} = g/n$ where m, n are chosen such that $\tilde{f}, \tilde{g}$ are primitive polynomials (the coefficients are integral with greatest common divisor 1).

Then, $\tilde{f}\tilde{g}$ is primitive by Gauss's Lemma, but this is also mnf_α which is primitive only when $mn = \pm 1$. This forces f, g to be primitive, hence integral. $\square$

We are now ready to prove our result.

Proof of Theorem 9.9: We may assume that $m, n \geq 1$; the case where G is a p-group can be dealt with by examining the class equation of G and concluding that $p \mid \operatorname{ord}(Z(G))$, whence $Z(G)$ is a non-trivial normal subgroup of G.

Let G be simple. This forces its center to be either $\{e\}$ or G; in the latter case, G is abelian! Thus, we have $Z(G) = \{e\}$. The class equation of G is thus of the form

$$\operatorname{ord}(G) = 1 + \sum_{\operatorname{ord}(\mathcal{O})>1} \operatorname{ord}(\mathcal{O}).$$

Now, pick $\mathcal{O}$ such that $q \nmid \operatorname{ord}(\mathcal{O})$. Using $\operatorname{ord}(\mathcal{O}) \mid \operatorname{ord}(G)$, we must have $\operatorname{ord}(\mathcal{O}) = p^\ell$ for some $1 \leq \ell \leq n$.

Let $\{\chi_i\}$ be the irreducible representations of G, where χ_1 is the trivial representation. The column orthogonality relations (Theorem 7.1) give $\sum_i \chi_i(1)\chi_i(\mathcal{O}) = 0$, whence

$$1 + \sum_{i>1} \chi_i(1)\chi_{i(\mathcal{O})} = 0.$$

It follows that there is a non-trivial irreducible character χ such that $\chi(\mathcal{O}) \neq 0$ and $p \nmid \chi(1)$. If not, then all $\chi_i(1)\chi_i(\mathcal{O})/p$ would be algebraic integers for $i > 1$, forcing $1/p$ to be an algebraic integer (dividing the previous equation by p), a contradiction.

Let σ be a representation corresponding to χ. Recall that each $e_{\mathcal{O}}$ is integral over $\mathbb{Z}$, so

$$\tilde{\sigma}(e_{\mathcal{O}}) = \frac{\operatorname{ord}(\mathcal{O}) \cdot \chi(\mathcal{O})}{\chi(1)}$$

is an algebraic integer. But $\operatorname{ord}(\mathcal{O})$ being a power of p is coprime with $\chi(1)$ by choice! Thus, $\chi(\mathcal{O})/\chi(1)$ is an algebraic integer too. To see this, use Bezout's Lemma to write $\alpha \operatorname{ord}(\mathcal{O}) + \beta\chi(1) = 1$ for integers α, β, and conclude that

$$\frac{\chi(\mathcal{O})}{\chi(1)} = \alpha \operatorname{ord}(\mathcal{O})\frac{\chi(\mathcal{O})}{\chi(1)} + \beta\chi(\mathcal{O}).$$

Recall that $\chi(\mathcal{O})$ is the sum of $d = \chi(1)$ roots of unity (Proposition 1.5), say $\chi(\mathcal{O}) = \xi_1 + \ldots + \xi_d$. Setting $\beta = \chi(\mathcal{O})/d$, it follows that the roots of the minimal polynomial f of β, which has integer coefficients, are the Galois conjugates of β; examine the field extensions $\mathbb{Q} \hookrightarrow \mathbb{Q}(\beta) \hookrightarrow K$ where $K/\mathbb{Q}$ is Galois. In other words, the roots of f are precisely $\{\tau(\beta) : \tau \in \operatorname{Gal}(K/\mathbb{Q})\}$. Since each of these $\tau(\beta) = \sum_i \tau(\xi_i)/d$, if $|\beta| < 1$, then $|\tau(\beta)| < 1$. Thus, the constant term of f, being the product of all such $\tau(\beta)$, must have absolute value < 1. This contradicts the fact that it must be a non-zero integer, forcing $|\beta| = 1$, hence all $\xi_i = \beta$ are equal.

With this, for any $s \in \mathcal{O}$, observe that $\sigma(s) = \beta \operatorname{id}_V$, hence it commutes with all $\sigma(g)$ for $g \in G$. Furthermore, σ is injective; if not, Proposition 9.10 would yield a contradiction. It follows that s commutes with all $g \in G$, whence $\mathcal{O} \subseteq Z(G)$, a contradiction! $\square$

9.3. Young tableaus

We are interested in calculating the irreducible representations of the symmetry group S_n. To do this, we must first understand its conjugacy classes; recall that they are characterized by *cycle types*. Thus, there is a one to one correspondence between the conjugacy classes of S_n and the integer partitions of n.

Definition 9.13 (Young diagrams and tableaus): Let $\lambda = (\lambda_1, ..., \lambda_k)$ be a partition of n, in the sense that $n = \lambda_1 + ... + \lambda_k$. Additionally, we impose $\lambda_1 \geq ... \geq \lambda_k$. The associated *Young diagram* for λ consists of n boxes arranged in a grid-like fashion, with precisely λ_i boxes in the i-th row. When the boxes are filled with distinct integers from $1, ..., n$, this is called a *Young tableau*. In particular, if the entries in each row and column appear in ascending order, we call this a *standard Young tableau*.

Example 9.13.1: Consider the partition $(3, 3, 1)$ of 7. Following are its Young diagram, a possible Young tableau of the same shape, as well as a standard Young tableau.

4	1	6
3	5	7
2		

1	2	6
3	5	7
4		

The group S_n acts naturally on a Young tableau by acting on its entries. We denote the image of a Young tableau T under the action of $g \in S_n$ simply by gT. If a box in T contained the integer a, then the same box in gT contains $g(a)$.

Definition 9.14: Let T be a Young tableau. We denote C_T as the elements of S_n stabilizing the columns of T, and R_T as the elements of S_n which stabilize the rows of T.

A permutation *stabilizes a column* of a Young tableau T if the image of each element remains in the same column, and similarly for stabilizing rows. It is clear that $R_T \cap C_T = \{1\}$.

Given a permutation λ, we let μ_i denote the number of boxes in the i-th column of the corresponding Young diagram.

Proposition 9.15: Let the Young diagram of λ have k rows and ℓ columns. Then,

$$R_T \cong S_{\lambda_1} \times ... \times S_{\lambda_k}, \qquad C_T \cong S_{\mu_1} \times ... \times S_{\mu_\ell}.$$

Lemma 9.16: Let T be a Young tableau, and let $g \in S_n$. Then,

$$R_{gT} = gR_Tg^{-1}, \qquad C_{gT} = gC_Tg^{-1}.$$

Proof: Fix a row i in T, and let its elements be $a_1, ..., a_{\lambda_i}$. Then, the i-th row in gT contains the elements $ga_1, ..., ga_{\lambda_i}$. Note that R_{gT} and gR_Tg^{-1} have the same size, so it is enough to show that for $h \in R_t$ we have $ghg^{-1} \in R_{gT}$. But ghg^{-1} sends the element ga_j of the i-th row in gT to $(ghg^{-1})ga_j = gha_j$. Furthermore, ha_j also belongs to the i-th row of T by choice of h, so $ha_j \in \{a_1, ..., a_{\lambda_i}\}$. Thus, ghg^{-1} sends $g(a_j)$ to $g(ha_j) \in \{g(a_1), ..., g(a_{\lambda_i})\}$ i.e. to the i-th row of gT, whence it stabilizes the i-th row of gT as desired. The proof for the second identity is similar. □

We now define three special elements of the group ring $\mathbb{C}[S_n]$.

Definition 9.17: Let T be a Young tableau. We denote

$$a_T = \sum_{g \in R_T} g, \qquad b_T = \sum_{g \in C_T} \operatorname{sign}(g)\, g, \qquad c_T = a_T \cdot b_T.$$

Proposition 9.18: Let T be a Young tableau, and let $h \in R_T$, $h' \in C_T$. Then,

$$h a_T = a_T, \qquad h' b_T = \operatorname{sign}(h')\, b_T, \qquad h c_T h' = \operatorname{sign}(h')\, c_T.$$

Lemma 9.19: Let T, T' be two Young tableaus such that there exist two integers which appear in the same row of T and in the same column of T'. Then, $a_T \cdot b_{T'} = 0 = b_{T'} \cdot a_T$.

Proof: Let p, q be two such integers, and let t be the transposition $(p\, q)$. Note that $t \in R_T \cap C_{T'}$. Write each coset from $R_T / \{1, t\}$ as $\{r, rt\}$ where r varies over the coset representatives $\mathcal{R}$, and rewrite

$$a_T = \sum_{r \in \mathcal{R}} r + \sum_{r \in \mathcal{R}} rt = \sum_{r \in R} r(1+t) = a'(1+t).$$

Thus,

$$a_T \cdot b_{T'} = a'(1+t) b_{T'} = a'(b_{T'} + \operatorname{sign}(t)\, b_{T'}) = 0.$$ □

Impose the lexicographic dictionary order on the collection of all partitions of S_n.

Definition 9.20: Let λ, μ be two partitions of S_n. We say that $\lambda > \mu$ if for some $1 \le i \le n$, we have $\lambda_i > \mu_i$ and $\lambda_j = \mu_j$ for all $j < i$.

Example 9.20.1: The partitions of 4 in increasing order are $(1,1,1,1)$, $(2,1,1)$, $(2,2)$, $(3,1)$, and (4).

Lemma 9.21: Let $\lambda > \mu$, and let T, T' be tableaus with shapes λ, μ respectively. Then, the conditions of Lemma 9.19 are satisfied, whence $a_T \cdot b_{T'} = 0 = b_{T'} \cdot a_T$.

Proof: Suppose not, whence any two numbers in the same row of T must belong to different columns of T'. This means that T' must have at least as many columns of as there are columns of T, so $\lambda_1 \le \mu_1$. The condition $\lambda > \mu$, forces $\lambda_1 = \mu_1$. Apply a suitable element of $C_{T'}$ to T' so that the elements from the first row of T all end up in the first row of T'. Now, delete the first rows of T, T', and repeat the previous argument until there are no rows remaining to show that $\lambda = \mu$. □

9.4. Irreducible representations of S_n

Returning to the problem of finding irreducible representations of S_n, note that $\mathbb{C}[S_n]$ is effectively the regular representation of S_n. Thus, it contains copies of every irreducible representation of S_n, via Proposition 5.1. This means that the search for irreducible representations of S_n reduces to finding S_n -invariant subspaces of $\mathbb{C}[S_n]$.

Observe that if $V \subseteq \mathbb{C}[S_n]$ is an irreducible representation of S_n, we may decompose $\mathbb{C}[S_n] = V \oplus W$ where W is another representation of S_n, via Maschke (Theorem 3.2). This gives us an S_n-invariant projection $\pi : \mathbb{C}[S_n] \to V$, which simply deletes the W component. Setting $\pi(1) = a$, note that

$$\pi(g) = \pi(g \cdot 1) = g \cdot \pi(1) = g \cdot a.$$

This shows that π is simply right multiplication by $\pi(1) = a$ in $\mathbb{C}[S_n]$. Also, $a \in V$, so $a = \pi(a) = \pi(1) \cdot a = a^2$. Thus, $V = \operatorname{im}(\pi) = \mathbb{C}[S_n] \cdot a$, where a is idempotent.

Proposition 9.22: Let $b \in \mathbb{C}[G]$. Then, $\mathbb{C}[G] \cdot b$ is G-invariant.

This motivates us to find idempotent elements $a \in \mathbb{C}[S_n]$ such that $\mathbb{C}[S_n] \cdot a$ is irreducible. Eventually, we will see that there are as many such elements as there are partitions of S_n.

Lemma 9.23: Let T, T' have the same shape. Then, $\mathbb{C}[S_n] \cdot c_T \cong \mathbb{C}[S_n] \cdot c_{T'}$ as representations of S_n.

Proof: Let g be the (unique) permutation from S_n such that $T = gT'$. It follows from Lemma 9.16 that

$$a_{T'} = g a_T g^{-1}, \qquad b_{T'} = g b_T g^{-1}, \qquad c_{T'} = g c_T g^{-1}.$$

Thus, $\mathbb{C}[S_n] \cdot c_{T'} = \mathbb{C}[S_n] \cdot g c_T g^{-1}$. Now, $\mathbb{C}[S_n] \cdot g = \mathbb{C}[S_n]$ since g merely permutes the basis elements, so $\mathbb{C}[S_n] \cdot c_{T'} = (\mathbb{C}[S_n] \cdot c_T)\, g^{-1}$. This means that the desired isomorphism is right multiplication by g, which is indeed an S_n-invariant map. □

Example 9.23.1: Consider the partition (n) of n.

Proposition 9.24: Let $H \subseteq G$, and let $a = \sum_{h \in H} h$. Then, $\dim(\mathbb{C}[G] \cdot a) = [G : H]$.

Indeed, the action of g sends a to $a_g = \sum_{h \in gH} h$. Thus, we have a bijection between the basis $\{a_g\}_{g \in G}$ of $\mathbb{C}[G] \cdot a$ and the cosets G/H, sending a_g to gH. Furthermore, this bijection is G-invariant: for $s \in G$, note that $s a_g = a_{sg}$. This gives us the following.

Proposition 9.25: Let $H \subseteq G$, let $a = \sum_{h \in H} h$, and let $\mathbf{1}_H$ denote the trivial representation of H. Then, $\mathbb{C}[G] \cdot a \cong \operatorname{Ind}_H^G(\mathbf{1}_H)$ as representations of G.

Lemma 9.26: Let $a \in \mathbb{C}[S_n]$. If $paq = \operatorname{sign}(q)\, a$ for all $p \in R_T$, $q \in C_T$, then a is a scalar multiple of c_T.

Proof: Write $a = \sum_{g\in G} \lambda_g g$, so $paq = \sum_{g\in G} \lambda_{p^{-1}gq^{-1}} g$. This forces all $\lambda_{pgq} = \operatorname{sign}(q)\,\lambda_g$ for all $g \in S_n$, $p \in R_T$, $q \in C_T$. Now,

$$c_T = \left(\sum_{p\in R_T} p\right)\left(\sum_{q\in C_T} \operatorname{sign}(q)\, q\right) = \sum_{\substack{p\in R_T\\ q\in C_T}} \operatorname{sign}(q)\, pq = \sum_{g\in G} \lambda'_g g,$$

where each $\lambda'_{pq} = \operatorname{sign}(q)$, $\lambda'_g = 0$ otherwise. Plugging in $g = 1$, we have $\lambda_{pq} = \operatorname{sign}(q)\,\lambda_1 = \lambda'_{pq}\lambda_1$, so it suffices to show that the remaining $\lambda_g = 0$ when $g \notin R_T C_T$.

Suppose that α, β appear in the same row of T and the same column of $T' = gT$. Then, setting $t = (\alpha\,\beta)$, we have $t \in R_T \cap C_{gT} = R_T \cap gC_Tg^{-1}$, whence $r = gsg^{-1}$ for some $s \in C_T$. Note that s must also be a transposition. Thus, $g = rgs$, so $\lambda_g = \lambda_{rgs} = \operatorname{sign}(s)\,\lambda_g = -\lambda_g$, forcing $\lambda_g = 0$.

With the above, it is enough to show that if no two such elements α, β exist, then $g \in R_T C_T$. Indeed, if all elements from the same row of T appear in distinct columns of $T' = gT$, then we may find $q' \in C_{T'}$ such that rows of T are the same as the rows of $q'T'$ as sets (following the argument of Lemma 9.21). Next, find $p \in R_T$ such that $pT = q'T'$. Finally, write $q' = gqg^{-1}$ for $q \in C_T$, whence $pT = gqT$, forcing $p = gq$. This shows that $g \in R_T C_T$ as desired. □

Remark: Compare this with Proposition 9.18.

Corollary 9.26.1: Let $a \in \mathbb{C}[S_n]$. Then, $c_T a c_T = \lambda_a c_T$.

We are now ready to prove our main results.

Theorem 9.27: Let T be a Young tableau. Then, $\mathbb{C}[S_n] \cdot c_T$ is an irreducible representation of S_n.

Proof: Set $V = \mathbb{C}[S_n] \cdot c_T$, and suppose that $W \subseteq V$ is S_n-invariant. We claim that W is either $\{0\}$ or V. The previous corollary gives $c_T V = c_T \cdot \mathbb{C}[S_n] \cdot c_T \subseteq \operatorname{span}_{\mathbb{C}}\{c_T\} = \mathbb{C}c_T$, so $c_T W \subseteq \mathbb{C}c_T$.

Now, if $c_T W = \mathbb{C}c_T$, then

$$V = \mathbb{C}[S_n] \cdot c_T = \mathbb{C}[S_n] \cdot \mathbb{C}c_T = \mathbb{C}[S_n] \cdot c_T W \subseteq \mathbb{C}[S_n] \cdot W \subseteq W,$$

where the final step follows from the S_n-invariance of W. This forces $V = W$.

Otherwise, if $c_T W = \{0\}$, recall that we may write $W = \mathbb{C}[S_n] \cdot a$ for some $a \in W$ such that $a^2 = a$. Then, for $v, w \in W$, we may write $v = b \cdot c_T \in \mathbb{C}[S_n] \cdot c_T = V$ for some $b \in \mathbb{C}[S_n]$, so that $vw = bc_T w = 0$. In particular, putting $v = w = a$ gives $a = a^2 = 0$, hence $W = \mathbb{C}[S_n] \cdot a = \{0\}$. □

Theorem 9.28: Let T, T' be two Young tableaus with different shapes. Then, $\mathbb{C}[S_n] \cdot c_T \ncong \mathbb{C}[S_n] \cdot c_{T'}$ as representations of S_n.

Proof: Set $V = \mathbb{C}[S_n] \cdot c_T$, $V' = \mathbb{C}[S_n] \cdot c_{T'}$, and suppose that $V \cong V'$. Without loss of generality, let $T > T'$, whence $a_T \cdot b_{T'} = 0$. Again, $c_T V \subseteq \mathbb{C}c_T$; if $c_T V = \{0\}$, then $V = \{0\}$ (via the same argument in the previous proof) hence so is V'.

Otherwise, $c_T V \neq \{0\}$. Observe that

$$c_T V' = a_T b_T \cdot \mathbb{C}[S_n] \cdot a_{T'} b_{T'} = a_T \cdot X \cdot b_{T'}.$$

We claim that this is 0; it is enough to show that $a_T g b_{T'} = 0$ for all $g \in S_n$. However, this is equivalent to $a_T b_{gT'} = 0$. Since T' and gT' have the same shape, this is equivalent to $a_T b_{T'} = 0$, which we already have from $T > T'$. □

Thus, we may enumerate all irreducible representations of S_n by picking a tableau T_λ of shape λ for each partition λ of n, then computing $\mathbb{C}[S_n] \cdot c_{T_\lambda}$.

9.5. Revisiting standard Young tableaus

Theorem 9.29: Let f_λ be the number of standard Young tableaus of shape λ. Then,

$$\sum_\lambda f_\lambda^2 = n!.$$

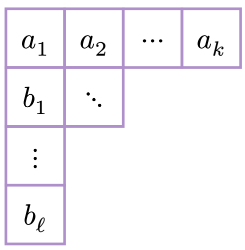
To prove the above, we will find a bijection between the collection of all ordered pairs of Young tableaus (P, Q) of the same shape, and all permutations in S_n. First, we must describe the process of *row insertion*. Given a tableau T with distinct positive entries and increasing rows and columns, and a positive integer k, we may obtain a new tableau T' (satisfying the same properties as T, with one more box), denoted $T \leftarrow k$ by inserting the element k as follows.

1. If possible, append k to the end of the first row of T, and stop.
2. Otherwise, find the rightmost element in the first row where k can be substituted, say k'. Replace k' with k.
3. Insert k' into next row, using the same process as above.

Proof of Theorem 9.29: Let $\sigma \in S_n$ be a permutation. Set $P = \sigma(1) \leftarrow \dots \leftarrow \sigma(n)$, i.e. successively insert the numbers $\sigma(1), \dots, \sigma(n)$. At each step, a new box is created. Construct a tableau Q according to the following rule: the integer i appears in a box in Q if the corresponding box in P was created at step i. This gives us a map $\sigma \mapsto (P, Q)$. It can be shown that this process is reversible! □

Definition 9.30 (Hook length): The hook length of a box in a Young diagram is the number of boxes in the L-shape formed by the box itself, the boxes to its right, and the boxes below it.

Consider a hook within a Young diagram of the following form, where each of the boxes is filled with its hook length.



For instance, $a_1 = k + \ell$. We must have $a_1 > a_2 > \dots > a_k$, and $a_1 > b_1 > \dots > b_\ell$.

Lemma 9.31: The numbers $a_1, \dots, a_k, a_1 - b_1, \dots, a_1 - b_\ell$ form a permutation of $1, \dots, k + \ell$.

Proof: It suffices to show that these numbers are distinct; the only possible duplicates are of the form $a_i = a_1 - b_j$, i.e. $a_1 = a_i + b_j$ for some i, j.

Suppose that the box which is both below a_i and to the right of b_j is not present in the diagram. Then, counting boxes gives $a_i \leq j + (k - i)$, and $b_j \leq (\ell - j) + (i - 1)$, which together give $k + \ell = a_1 = a_i + b_j \leq k + \ell - 1$, a contradiction.

Similarly, when the box below a_i and to the right of b_j is present, we will obtain $k + \ell = a_i + b_j \geq k + \ell + 1$, another contradiction. □