MA4203: Probability II

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Assignment I

Exercise 1 Let $r > 0$ and let X be any almost surely non-negative random variable. Prove the following.

(a)
$$E(X) = \int_0^\infty P(X > x)\, dx.$$

(b)
$$\sum_{n=1}^\infty P(X \geq n) \leq E(X) \leq 1 + \sum_{n=1}^\infty P(X \geq n).$$

(c)
$$E(X^r) = r\int_0^\infty x^{r-1} P(X > x)\, dx.$$

(d)
$$A\sum_{n=1}^\infty n^{r-1} P(X \geq n) \leq E(X^r) \leq 1 + B\sum_{n=1}^\infty n^{r-1} P(X \geq n).$$

(e) Let g be a non-negative strictly increasing differentiable function. Then,
$$E(g(X)) = g(0) + \int_0^\infty g'(x) P(X > x)\, dx.$$
Hence,
$$E(g(X)) < \infty \iff \sum_{n=1}^\infty g'(n) P(X > n) < \infty.$$

(f)
$$E(\log^+ X) < \infty \iff \sum_{n=1}^\infty n^{-1} P(X > n) < \infty.$$

(g) For $p > 1$,
$$E((\log^+ X)^p) < \infty \iff \sum_{n=1}^\infty n^{-1} (\log x)^{p-1} P(X > n) < \infty.$$

(h) For $p > 0$ and $r > 1$,
$$E(X^r (\log^+ X)^p) < \infty \iff \sum_{n=1}^\infty n^{r-1} (\log x)^p P(X > n) < \infty.$$

(i)
$$E(\log^+ \log^+ X) < \infty \iff \sum_{n=1}^\infty n^{-1} (\log n)^{-1} P(X > n) < \infty.$$

Solution.

(a) Let F be the cdf of X. Then, Tonelli's Theorem gives

$$\begin{aligned}
E(X) = \int_0^\infty x\, dF(x) &= \int_0^\infty \int_0^x dt\, dF(x) \\
&= \int_0^\infty \int_0^\infty \chi_{(0,x)}(t)\, dt\, dF(x) \\
&= \int_0^\infty \int_0^\infty \chi_{(t,\infty)}(x)\, dF(x)\, dt \\
&= \int_0^\infty 1 - F(t)\, dt \\
&= \int_0^\infty P(X > t)\, dt.
\end{aligned}$$

(b) Note that if $n \le x < n+1$, then

$$P(X < n) \le P(X \le x) \le P(X < n+1), \qquad P(X \ge n+1) \le P(X > x) \le P(X \ge n).$$

Integrating from n to $n+1$ and summing, we have

$$\sum_{n=0}^\infty P(X \ge n+1) \le E(X) \le \sum_{n=0}^\infty P(X \ge n) = P(X \ge 0) + \sum_{n=1}^\infty P(X \ge n).$$

Thus,

$$\sum_{n=1}^\infty P(X \ge n) \le E(X) \le 1 + \sum_{n=1}^\infty P(X \ge n).$$

(c) Using the substitution $x \mapsto u^r$,

$$E(X^r) = \int_0^\infty P(X^r > x)\, dx = \int_0^\infty P(X > x^{1/r})\, dx = \int_0^\infty P(X > u)\, r u^{r-1}\, du.$$

(d) Check that each

$$rn^{r-1} < (n+1)^r - n^r < r(n+1)^{r-1}.$$

To see this, note that $(n+1)^r - n^r = rx^{r-1}$ for some $x \in (n, n+1)$ by the Mean Value Theorem. Furthermore,

$$(n+1)^{r-1} = \left(\frac{n+1}{n}\right)^{r-1} n^{r-1} \le r2^{r-1} n^{r-1},$$

and

$$rn^{r-1} = \left(\frac{n}{n+1}\right)^{r-1} (n+1)^{r-1} \ge r2^{-(r-1)}(n+1)^{r-1}.$$

Thus, we have

$$r2^{-(r-1)}(n+1)^{r-1} < (n+1)^r - n^r < r2^{r-1} n^{r-1}.$$

Now,

$$E(X^r) = \int_0^\infty r x^{r-1} P(X > x)\, dx = \sum_{n=0}^\infty \int_n^{n+1} r x^{r-1} P(X > x)\, dx.$$

Thus,

$$\sum_{n=0}^\infty P(X \ge n+1) \int_n^{n+1} r x^{r-1}\, dx \le E(X^r) \le \sum_{n=0}^\infty P(X \ge n) \int_n^{n+1} r x^{r-1}\, dx,$$

so

$$\sum_{n=0}^\infty P(X \ge n+1)[(n+1)^r - n^r] \le E(X^r) \le \sum_{n=0}^\infty P(X \ge n)[(n+1)^r - n^r].$$

Using the established inequality and re-indexing,

$$\sum_{n=0}^{\infty} P(X \geq n+1) \cdot r2^{-(r-1)}(n+1)^{r-1} \leq E(X^r) \leq 1 + \sum_{n=1}^{\infty} P(X \geq n) \cdot r2^{r-1}n^{r-1},$$

$$r2^{-(r-1)} \sum_{n=1}^{\infty} n^{r-1} P(X \geq n) \leq E(X^r) \leq 1 + r2^{r-1} \sum_{n=1}^{\infty} n^{r-1} P(X \geq n).$$

(e) Calculate

$$\begin{aligned} E(g(X)) &= \int_0^\infty g(x)\, dF(x) \\ &= \int_0^\infty (g(0) + g(x) - g(0))\, dF(x) \\ &= g(0) + \int_0^\infty \int_0^x g'(t)\, dt\, dF(x) \\ &= g(0) + \int_0^\infty \int_t^\infty g'(t)\, dF(x)\, dt \\ &= g(0) + \int_0^\infty g'(t)(1 - F(t))\, dt \\ &= g(0) + \int_0^\infty g'(t) P(X > t)\, dt. \end{aligned}$$

We now impose the condition that there exist constants A, B such that for $x \in [n, n+1]$, sufficiently large n,

$$A\, g'(n+1) \leq g'(x) \leq B\, g'(n).$$

Then, $E(g(X)) < \infty$ if and only if the following integral is finite.

$$\int_0^\infty g'(x) P(X > x)\, dx = \sum_{n=0}^{\infty} \int_n^{n+1} g'(x) P(X > x)\, dx.$$

Now,

$$\sum_{n=0}^{\infty} A g'(n+1) P(X > n+1) \leq \sum_{n=0}^{\infty} \int_n^{n+1} g'(x) P(X > x)\, dx \leq \sum_{n=0}^{\infty} B g'(n) P(X > n),$$

hence

$$A \sum_{n=1}^{\infty} g'(n) P(X > n) \leq \int_0^\infty g'(x) P(X > x)\, dx \leq B g'(0) + B \sum_{n=1}^{\infty} g'(n) P(X > n).$$

Thus, $E(g(X)) < \infty$ if and only if $\sum_{n=1}^{\infty} g'(n) P(X > n) < \infty$.

Remark. Instead of starting at 0, we may have to truncate the integral/sum.

(f) Note that

$$E(g(X)) = \int_0^\infty g(x)\, dF(x) = \int_0^M g(x)\, dF(x) + \int_M^\infty g(x)\, dF(x),$$

the first part being finite, so we need only check the criterion for g and for the sum described in the previous part on (M, ∞) and for $n > M$. Here, $g = \log^+$, which is non-negative, strictly increasing, and differentiable on $(1, \infty)$, with $g'(x) = 1/x$. Now, for $x \in [n, n+1]$, we have

$$\frac{1}{n+1} \leq \frac{1}{x} \leq \frac{1}{n}.$$

With this, the desired follows via the criterion in the previous part.

(g) Putting $g = (\log^+)^p$, we have g non-negative, strictly increasing, differentiable on $(2, \infty)$, with $g'(x) = px^{-1}(\log x)^{p-1}$. For $x \in [n, n+1]$, we have

$$\log x \leq \log(n+1) < \log(n^2) = 2\log n,$$

and

$$\log x > \log n = \frac{1}{2}\log n^2 > \frac{1}{2}\log(n+1).$$

Thus,

$$2^{-(p-1)}(n+1)^{-1}(\log(n+1))^{p-1} \leq x^{-1}(\log x)^{p-1} \leq 2^{p-1}n^{-1}(\log n)^{p-1}.$$

Again, the desired criterion follows.

(h) Precisely the same calculations as in (c) show that if there exist constants A, B and a map $h\colon \mathbb{N} \to \mathbb{R}$ such that for all $x \in [n, n+1]$,

$$A\,h(n+1) \leq g'(x) \leq B\,h(n),$$

then

$$E(g(x)) < \infty \iff \sum_{n=1}^{\infty} h(n)P(X > n) < \infty.$$

Putting $g(x) = x^r(\log^+ x)^p$, we have g non-negative, strictly increasing, differentiable on $(1, \infty)$, with $g'(x) = x^{r-1}(\log x)^{p-1}[r\log x + 1]$. For sufficiently large n

$$r\log x \leq r\log x + 1 \leq 2r\log x, \qquad rx^{r-1}(\log x)^p \leq g'(x) \leq 2rx^{r-1}(\log x)^p.$$

for $x \in [n, n+1]$. Thus,

$$r \cdot 2^{-(r-1)}(n+1)^{r-1} \cdot 2^{-p}(\log(n+1))^p \leq g'(x) \leq 2r \cdot 2^{r-1}n^{r-1} \cdot 2^p(\log n)^p.$$

Thus, putting $h(n) = n^{r-1}(\log n)^p$ gives the desired criterion.

(i) Putting $g = \log^+\log^+$, we have g non-negative, strictly increasing, differentiable on (e, ∞), with $g'(x) = x^{-1}(\log x)^{-1}$. Also, for $x \in [n, n+1]$, we have

$$(n+1)^{-1}(\log(n+1))^{-1} \leq g'(x) \leq n^{-1}(\log n)^{-1}.$$

The desired criterion follows.

Exercise 2 Let $\{X_n\}$ be a sequence of pairwise independent and identically distributed random variables.

(a) Show that $X_n/n \to 0$ almost surely if and only if $E(|X_1|) < \infty$.

(b) Show that $|X_n|^{1/n} \to 1$ almost surely if and only if $E(\log^+|X_1|) < \infty$.

Solution.

(a)

$$\begin{aligned}
X_n/n \xrightarrow{as} 0 &\iff \sum_{n=1}^{\infty} P(|X_n/n| > \epsilon) < \infty \text{ for all } \epsilon > 0 && \text{(Borel-Cantelli)}\\
&\iff \sum_{n=1}^{\infty} P(|X_n/\epsilon| > n) < \infty \text{ for all } \epsilon > 0 \\
&\iff \sum_{n=1}^{\infty} P(|X_1/\epsilon| > n) < \infty \text{ for all } \epsilon > 0 && \text{(Identical distributions)}\\
&\iff E(|X_1/\epsilon|) < \infty \text{ for all } \epsilon > 0 \\
&\iff E(|X_1|) < \infty.
\end{aligned}$$

Remark. Pairwise independence is only needed for the forward implication in the first step.

(b) Note that $(\log^+ |X_n|)/n \xrightarrow{as} 0$ if and only if $E(\log^+ |X_1|) < \infty$ by the previous part. By the continuous mapping theorem, $|X_n|^{1/n} \xrightarrow{as} 1$ gives $(\log^+ |X_n|)/n \xrightarrow{as} 0$. Conversely, $(\log^+ |X_n|)/n \xrightarrow{as} 0$ means that $(\log^+ |X_n(\omega)|)/n \to 0$ for all $\omega \in A$ with $P(A = 1)$. Thus, $|X_n(\omega)|/n \to 0$ for those $\omega \in A$ where $|X_n(\omega)| > 1$; for the remaining ω, we already have $|X_n(\omega)|/n \leq 1/n \to 0$.

Exercise 3 Let $\{X_n\}$ be a sequence of identically distributed variables, and let $M_n = \max\{|X_1|, \dots, |X_n|\}$.

(a) If $E(|X_1|) < \infty$, show that $M_n/n \to 0$ almost surely.

(b) If $E(|X_1|^\alpha) < \infty$ for some $\alpha \in (0, \infty)$, then show that $M_n/n^{1/\alpha} \to 0$ almost surely.

(c) If $\{X_n\}$ is also independent, then show that $M_n/n^{1/\alpha} \to 0$ almost surely implies that $E(|X_1|^\alpha) < \infty$.

Solution.

(a) We have $E(|X_1|) < \infty \implies X_n/n \xrightarrow{as} 0$, hence there exists $A \subseteq \Omega$ with $P(A) = 1$ such that $X_n(\omega)/n \to 0$ for all $\omega \in A$. Set $x_n = X_n(\omega)$, $m_n = M_n(\omega)$, and let $\epsilon > 0$, whence there exists $N \in \mathbb{N}$ such that for all $n > N$, we have $|x_n|/n < \epsilon$. Now, observe that for all $n > N$,

$$\frac{m_n}{n} \leq \frac{\max\{|x_1|, \dots, |x_N|\}}{n} + \max\left\{\frac{|x_{N+1}|}{N+1}, \dots, \frac{|x_n|}{n}\right\}.$$

This is because either $m_n \in \{|x_1|, \dots, |x_N|\}$, or $m_n = |x_k|$ for some $N < k \leq n$, so $m_n/n \leq |x_k|/k$. Note that as $n \to \infty$, the first term vanishes and the second term is always bounded by ϵ. Thus, $m_n/n \to 0$, hence $M_n/n \xrightarrow{as} 0$.

(b) We have $E(|X_1|^\alpha) < \infty \implies M_n^\alpha/n \xrightarrow{as} 0$. By the continuous mapping theorem, $M_n/n^{1/\alpha} \xrightarrow{as} 0$.

(c) We have $M_n/n^{1/\alpha} \xrightarrow{as} 0 \implies |X_n|/n^{1/\alpha} \xrightarrow{as} 0$, as $|X_n| \leq M_n$. By the continuous mapping theorem, $|X_n|^\alpha/n \xrightarrow{as} 0$. Thus, the previous exercise gives $E(|X_1|^\alpha) < \infty$.

Exercise 4 Let $\{A_n\}$ be a sequence of independent events with all $P(A_n) < 1$. Show that

$$P(\limsup_{n\to\infty} A_n) = 1 \iff P\left(\bigcup_{n=1}^\infty A_n\right) = 1.$$

Solution. We have

$$P(\limsup_{n\to\infty} A_n) = 1 \iff P\left(\bigcap_{k=1}^\infty \bigcup_{n=k}^\infty A_n\right) = 1 \implies P\left(\bigcup_{n=1}^\infty A_n\right) = 1.$$

For the reverse implication, it suffices to show that for all $k \geq 1$, we have

$$P\left(\bigcup_{n=k}^\infty A_n\right) = 1,$$

which is equivalent to

$$P\left(\bigcap_{n=k}^\infty A_n^c\right) = 0 \iff \prod_{n=k}^\infty P(A_n^c) = 0.$$

But this follows immediately since it holds for $k = 1$; each $P(A_n^c) > 0$ means that this term can be cancelled from the product yielding the equality for all $k > 1$.

Exercise 5 Let $\{X_n\}$ be a sequence of independent random variables such that for $n \geq 1$, we have $P(X_n = 1) = n^{-1} = 1 - P(X_n = 0)$. Show that X_n converges to 0 in probability but not almost surely.

Solution. Note that for $\epsilon > 0$, we have $P(|X_n| > \epsilon) \leq 1/n \to 0$. Thus, $X_n \xrightarrow{p} 0$.

By the Borel-Cantelli Lemmas, $X_n \xrightarrow{as} 0$ is equivalent to $\sum_{n=1}^{\infty} P(|X_n| > \epsilon) < \infty$ for all $\epsilon > 0$. However, this is false; putting $\epsilon = 1/2$, the latter sum is $\sum_{n=1}^{\infty} 1/n = \infty$.

Exercise 6 Let $\{X_n\}$ be a sequence of independent and identically distributed exponential random variables with density $f(x) = e^{-x}\chi_{(0,\infty)}(x)$. Define

$$Y_n = \max_{1 \leq i \leq n} X_i.$$

(a) Show that

$$\sum_{n=1}^{\infty} P(X_n > \epsilon \log n)$$

converges if $\epsilon > 1$ and diverges if $0 < \epsilon \leq 1$.

(b) Show that $\limsup X_n / \log n = 1$ almost surely.

(c) Show that $\liminf X_n / \log n = 0$ almost surely.

(d) Show that $[X_n > \epsilon \log n \text{ i.o.}] \iff [Y_n > \epsilon \log n \text{ i.o.}]$, and hence $\limsup Y_n/n = 1$ almost surely.

Solution.

(a) Calculate

$$P(X_n > \epsilon \log n) = \int_{\epsilon \log n}^{\infty} e^{-x}\, dx = \frac{1}{n^{\epsilon}}.$$

Thus

$$\sum_{n=1}^{\infty} P(X_n > \epsilon \log n) = \sum_{n=1}^{\infty} \frac{1}{n^{\epsilon}}$$

converges precisely when $\epsilon > 1$, and diverges when $0 < \epsilon \leq 1$.

(b) Putting $\epsilon = 1$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} P(X_n > \log n) = \infty &\iff P(X_n/\log n > 1 \text{ i.o.}) = 1 && \text{(Borel-Cantelli)} \\ &\implies P(\limsup X_n/\log n \geq 1) = 1 \\ &\iff \limsup X_n/\log n = 1 \text{ almost surely.} \end{aligned}$$

The second implication follows from the fact that there is a probability one set A such that for $\omega \in A$, we have $X_n(\omega)/\log n > 1$ infinitely often, i.e. there is a subsequence $\{n_k\}$ such that all $X_{n_k}(\omega)/\log n_k > 1$.

Next, for $\epsilon > 1$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} P(X_n > \epsilon \log n) < \infty &\iff P(X_n/\log n > \epsilon \text{ i.o.}) = 0 && \text{(Borel-Cantelli)} \\ &\iff P(X_n/\log n \leq \epsilon \text{ eventually}) = 1 \\ &\implies P(\limsup X_n/\log n \leq \epsilon) = 1 \\ &\iff \limsup X_n/\log n \leq \epsilon \text{ almost surely.} \end{aligned}$$

The third implication follows from the fact that there is a probability one set A_ϵ such that for $\omega \in A_\epsilon$, we have $X_{n_k}(\omega)/\log n_k \leq \epsilon$ for all sufficiently large n_k, for all subsequences $\{n_k\}$. Since all $A_{1+1/k}$ have probability one, their intersection has probability one by continuity from above, yielding

$$\limsup X_n/\log n \le 1 \text{ almost surely.}$$

Combining the two parts gives the desired result.

(c) Note that for all $0 < \epsilon < 1$,

$$\sum_{n=1}^{\infty} P(X_n < \epsilon \log n) = \sum_{n=1}^{\infty} 1 - \frac{1}{\epsilon} = \infty \iff P(X_n/\log n < \epsilon \text{ i.o.}) = 1$$
$$\implies P(\liminf X_n/\log n \le \epsilon) = 1.$$

Putting $\epsilon = 1/k \to 0$, continuity from above gives

$$\liminf X_n/\log n = 0.$$

(d) It is equivalent to show that

$$A = [X_n \le \epsilon \log n \text{ eventually}] \iff [Y_n \le \epsilon \log n \text{ eventually}] = B.$$

Since $X_n \le Y_n$, we have $\omega \in B \implies Y_n(\omega) \le \epsilon \log n$ for $n \ge N_\epsilon$, hence $X_n(\omega \le \epsilon \log n)$ for $n \ge N_\epsilon$ $\implies \omega \in A$.

Next, let $\omega \in A$, and let $N \in \mathbb{N}$ such that for all $n > N$, we have $X_n(\omega) \le \epsilon \log n$. Then,

$$\frac{Y_n(\omega)}{\log n} \le \frac{\max\{X_1(\omega), \dots, X_N(\omega)\}}{\log n} + \max\left\{\frac{X_{N+1}(\omega)}{\log(N+1)}, \dots, \frac{X_n(\omega)}{\log n}\right\}.$$

The second term is bounded by ϵ, while the first vanishes as $n \to \infty$. Thus, $Y_n(\omega)/\log n \le 2\epsilon$ eventually, so $\omega \in B$.

We have shown that $A = B$; repeating the proof of (b) with $X_n > \epsilon \log n$ i.o. replaced with $Y_n > \epsilon \log n$ i.o. proves that $\limsup Y_n/\log n = 1$ almost surely.

Exercise 7 Show that for any given sequence of random variables $\{X_n\}$, there exists a (deterministic) real sequence $\{a_n\}$ such that $X_n/a_n \xrightarrow{as} 0$.

Solution. Note that for any fixed X_n, we have $P(|X_n| > M) \to 0$ as $M \to \infty$. Thus, for each $n \in \mathbb{N}$, we can pick numbers M_n such that

$$P(|X_n| > M_n) < \frac{1}{2^n}.$$

Set $a_n = nM_n$. Then for $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n/a_n| > \epsilon) = \sum_{n=1}^{\infty} P(|X_n| > \epsilon n M_n).$$

Let $N \in \mathbb{N}$ such that $N\epsilon > 1$. Then the tail of the above series is

$$\sum_{n=N}^{\infty} P(|X_n| > \epsilon n M_n) \le \sum_{n=N}^{\infty} P(|X_n| > M_n) = \sum_{n=N}^{\infty} \frac{1}{2^n} < \infty.$$

Exercise 8 Let $\{X_n\}$ be a sequence of independent random variables such that

$$P(X_n = 2) = P(X_n = n^\beta) = a_n, \qquad P(X_n = a_n) = 1 - 2a_n,$$

for some $a_n \in (0, 1/3)$ and $\beta \in \mathbb{R}$. Show that $\sum_{n=1}^{\infty} X_n$ converges if and only if $\sum_{n=1}^{\infty} a_n < \infty$.

Solution. Define $Y_n = X_n \chi_{[|X_n| \le A]}$ for $A > 0$. First, suppose that $\beta > 0$, whence $n^\beta \to \infty$. Thus, putting $A = 3$, for sufficiently large n, we have

$$E(Y_n) = 2a_n + a_n(1 - 2a_n) = 3a_n - 2a_n^2 < 3a_n,$$

using $a_n^2 \geq 0$. Also,

$$V(Y_n) < E(Y_n^2) = 4a_n + a_n^2(1-2a_n) = 4a_n - a_n^2 + 2a_n^3 < 6a_n,$$

using $a_n^3 < a_n$. Finally,

$$P(|X_n| > 3) = a_n.$$

Thus, if $\sum_{n=1}^{\infty} a_n < \infty$, we have $\sum_{n=1}^{\infty} P(|X_n| > 3) < \infty$, $\sum_{n=1}^{\infty} E(Y_n) < \infty$, $\sum_{n=1}^{\infty} V(Y_n) < \infty$, whence Kolmogorov's Three Series Theorem gives the convergence of $\sum_{n=1}^{\infty} X_n$. Conversely, the convergence of $\sum_{n=1}^{\infty} X_n$ gives the convergence of $\sum_{n=N}^{\infty} P(|X_n| > 3) = \sum_{n=N}^{\infty} a_n$.

Next, suppose that $\beta \leq 0$. Thus, putting $A = 1$, for sufficiently large n, we have $n^\beta \leq 1$, hence

$$E(Y_n) = n^\beta a_n + a_n(1-2a_n) = n^\beta a_n + a_n - 2a_n^2 < 2a_n.$$

Also,

$$V(Y_n) < E(Y_n^2) = n^{2\beta} a_n + a_n^2(1-2a_n) = n^{2\beta} a_n + a_n^2 - 2a_n^3 < 2a_n,$$

using $a_n^2 < a_n$. Finally,

$$P(|X_n| > 1) = a_n.$$

Using precisely the same argument as before, $\sum_{n=1}^{\infty} X_n$ converges almost surely if and only if $\sum_{n=1}^{\infty} a_n$ converges.

Exercise 9 Let $\{X_n\}$ be a sequence of independent random variables such that $E(X_n) = 0$, $E(X_n^2) = \sigma^2$. Define $s_n^2 = \sum_{k=1}^{n} \sigma_k^2 \to \infty$. Show that for any $a > 1/2$,

$$Y_n = \frac{1}{s_n (\log s_n^2)^a} \sum_{k=1}^{n} X_k \xrightarrow{as} 0.$$

Solution. Set

$$Z_n = \frac{X_n}{s_n (\log s_n^2)^a}.$$

Without loss of generality, let $s_1 > 1$. Then,

$$\begin{aligned}
\sum_{n=1}^{\infty} V(Z_n) &= \sum_{n=1}^{\infty} \frac{\sigma_n^2}{s_n^2 (\log s_n^2)^{2a}} \\
&= \sum_{n=1}^{\infty} \frac{s_n^2 - s_{n-1}^2}{s_n^2 (\log s_n^2)^{2a}} \\
&\leq \sum_{n=1}^{\infty} \int_{s_{n-1}^n}^{s_n^2} \frac{dx}{x (\log x)^{2a}} \\
&= \sum_{n=1}^{\infty} - \frac{1}{(2a-1)\log(x)^{2a-1}} \Big|_{s_{n-1}^2}^{s_n^2} \\
&= \frac{1}{2a-1} \sum_{n=1}^{\infty} \frac{1}{(\log s_{n-1}^2)^{2a-1}} - \frac{1}{(\log s_n^2)^{2a-1}} \; < \; \infty.
\end{aligned}$$

Thus, $\sum_{n=1}^{\infty} Z_n$ converges in L_2, hence in probability, hence almost surely by Levy's Theorem. Finally, Kronecker's Lemma gives

$$Y_n = \frac{1}{s_n (\log s_n^2)^a} \sum_{k=1}^{n} Z_k s_k (\log s_k^2)^a \xrightarrow{as} 0.$$

Exercise 10 Let f be a bounded measurable function on $[0, 1]$ that is continuous at $1/2$. Evaluate

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 f\left(\frac{x_1 + \cdots + x_n}{n}\right) dx_1\, dx_2\, \cdots\, dx_n.$$

Solution. Let U_n be independent and identically distributed random variables, having uniform distribution on $[0, 1]$. Then, the given integral is simply $E(f(\bar{U}_n))$, where $\bar{U}_n = (U_1 + \cdots + U_n)/n$. Since $E(|U_1|) < \infty$, Kolmogorov's Strong Law of Large Numbers gives $\bar{U}_n \to E(U_1) = 1/2$. Since f is continuous at $1/2$, we have $f(\bar{U}_n) \to f(1/2)$. Finally, f is bounded, so the desired limit is $f(1/2)$ by Lebesgue's dominated convergence theorem.

Exercise 11 Let $\{X_n\}$ be a sequence of independent and identically distributed random variables. Investigate the almost sure convergence/divergence of the series $\sum_{n=1}^{\infty} X_n$.

Solution. Kolmogorov's Three-Series Theorem says that if $\sum_{n=1}^{\infty} X_n$ converges almost surely, then for all $A > 0$,

$$\sum_{n=1}^{\infty} P(|X_1| > A) = \sum_{n=1}^{\infty} P(|X_n| > A) < \infty.$$

This sum is finite precisely when $P(|X_1| > A) = 0$ for all $A > 0$. Thus, the series $\sum_{n=1}^{\infty} X_n$ converges almost surely only when each X_n is a degenerate random variable with $P(X_n = 0) = 1$.

Exercise 12 Let $\{X_n\}$ be a sequence of independent random variables with $E(X_n) = 0$ and $E(X_n^2) = \sigma_n^2 < \infty$. Suppose that $\sum_{n=1}^{\infty} \sigma_n^2/b_n^2 < \infty$ for some positive sequence $\{b_n\}$ which increases to ∞. Show that $b_n^{-1} \sum_{i=1}^{n} X_i \xrightarrow{as} 0$.

Solution. By Kronecker's Lemma, it is enough to show that $\sum_{n=1}^{\infty} X_n/b_n$ converges almost surely. By Levy's Theorem, it is equivalent to show that this converges in probability. This it is enough to show convergence in L^2, whence is it enough to show that the sum of variances $\sum_{n=1}^{\infty} V(X_n/b_n)$ is finite. This is precisely $\sum_{n=1}^{\infty} \sigma_n^2/b_n^2 < \infty$ as given.

Exercise 13 Let $\{X_n\}$ be a sequence of independent random variables with each $X_n \sim N(\mu_n, \sigma_n^2)$. Show that $\sum_{n=1}^{\infty} X_n$ converges almost surely if and only if both $\sum_{n=1}^{\infty} \mu_n$ and $\sum_{n=1}^{\infty} \sigma_n^2$ converge.

Solution. ($\Leftarrow$) We have $\sum_{n=1}^{\infty} V(X_n - \mu_n) = \sum_{n=1}^{\infty} \sigma_n^2 < \infty$, hence $\sum_{n=1}^{\infty}(X_n - \mu_n)$ converges almost surely. Using $\sum_{n=1}^{\infty} \mu_n < \infty$, we have $\sum_{n=1}^{\infty} X_n < \infty$ almost surely.

($\Rightarrow$) First, suppose that all $\mu_n = 0$. Since $\sum_{n=1}^{\infty} X_n$ converges almost surely, Kolmogorov's Three-Series Theorem gives

$$\sum_{n=1}^{\infty} P(|X_n/\sigma_n| > A/\sigma_n) = \sum_{n=1}^{\infty} P(|X_n| > A) < \infty$$

for all $A > 0$; fix $A = 1$. Now, $Z_n = X_n/\sigma_n$ are independent and identically distributed standard normal random variables. Thus, $P(|Z_n| > A/\sigma_n) \to 0$ forces $\sigma_n \to 0$. We also have

$$\sum_{n=1}^{\infty} \sigma_n^2 E(Z_n^2 \chi_{|Z_n| \le A/\sigma_n}) = \sum_{n=1}^{\infty} E(X_n^2 \chi_{|X_n| \le A}) = \sum_{n=1}^{\infty} V(X_n \chi_{|X_n| \le A}) < \infty$$

Now, $A/\sigma_n \to \infty$, so there exists $N \in \mathbb{N}$ such that $A/\sigma_n \ge 1$ for all $n \ge N$. Then,

$$E(Z_n^2 \chi_{|Z_n| \le A/\sigma_n}) \ge E(Z_n^2 \chi_{|Z_n| \le 1}) = K > 0.$$

This immediately gives

$$\sum_{n=N}^{\infty} K\sigma_n^2 \le \sum_{n=N}^{\infty} \sigma_n^2 E(Z_n^2 \chi_{|Z_n| \ge A/\sigma_n}) < \infty,$$

hence $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$.

Returning to the general case, write

$$S_n = \sum_{i=1}^{n} X_i \sim N(m_n, s_n^2), \qquad m_n = \sum_{i=1}^{n} \mu_i, \qquad s_n^2 = \sum_{i=1}^{n} \sigma_i^2.$$

Since S_n converges almost surely to some random variable Y, it also does so in distribution, hence the characteristic functions $\phi_{S_n}(t) \to \phi_Y(t)$ for all $t \in \mathbb{R}$. Now,

$$\phi_{S_n}(t) = e^{itm_n - t^2 s_n^2/2}, \qquad e^{-t^2 s_n^2/2} = |\phi_{S_n}(t)| \to |\phi_Y(t)|.$$

Since $\phi_Y(0) = 1$, the continuity of ϕ_Y gives $\phi_Y > 0$ on some neighbourhood of 0. There, applying logarithms gives

$$-\frac{1}{2}t^2 s_n^2 \to \log|\phi_Y(t)|.$$

Fixing one such $t \neq 0$, we have $s_n^2 \to -2\log|\phi_Y(t)|/t^2$, hence $\sum_{n=1}^{\infty} \sigma_n^2$ converges, say to s^2. Thus,

$$e^{itm_n} \to e^{t^2 s^2/2}\phi_Y(t)$$

for all $t \in \mathbb{R}$. This forces the convergence of m_n, hence of $\sum_{n=1}^{\infty} \mu_n$.

To justify the last step, suppose that the sequences $\{e^{-it\alpha_n}\}$ converge for all $t \in \mathbb{R}$. Then, we have each $e^{it(\alpha_n - \alpha_m)} \to 1$. It is enough to show that $\alpha_n - \alpha_m \to 0$, whence $\{\alpha_n\}$ is Cauchy, hence convergent. Thus, suppose that $e^{it\beta_k} \to 1$ for all $t \in \mathbb{R}$. Also suppose that all $\beta_k \neq 0$; we show that $\beta_k \to 0$ (we can re-insert any 0 values after initially removing them). Then, the dominated convergence theorem gives

$$\int_0^1 e^{it\beta_k}\, dt \to \int_0^1 dt = 1,$$

while

$$\int_0^1 e^{it\beta_k}\, dt = \frac{1}{i\beta_k}(e^{i\beta_k} - 1).$$

Thus, the following limit exists, and is given by

$$\lim_{n\to\infty} \beta_k = \lim_{n\to\infty} -i(e^{i\beta_k} - 1) = 0.$$

Assignment II

Exercise 1 Suppose that $\{x_n\}_{n\geq 1}$ is a sequence of real numbers. Define a sequence of random variables $\{X_n\}_{n\geq 1}$ as follows: $P(X_n = x_n) = 1$ for each $n \geq 1$.

(a) Show that if $x_n \to x$, then X_n converges weakly to X, where $P(X = x) = 1$.

(b) Show that if X_n converges weakly to X as above, then $x_n \to x$.

(c) Show that if X_n converges weakly to a random variable Y, then $P(Y = x) = 1$ for some $x \in \mathbb{R}$.

Solution.

(a) Note that for any continuous bounded function f, we have

$$E(f(X_n)) = f(x_n) \to f(x) = E(f(X)),$$

hence $X_n \xrightarrow{d} X$.

(b) Note that for all $y \neq x$, we have $F_{X_n}(y) \to F_X(y)$. This means that $F_{X_n}(y) \to 0$ when $y < x$, and $F_{X_n}(y) \to 1$ when $y > x$. But F_{X_n} only takes values in $\{0, 1\}$; thus, for any $\epsilon > 0$, there exists sufficiently large N such that $F_{X_n}(x - \epsilon) = 0$ and $F_{X_n}(x + \epsilon) = 1$ for all $n \geq N$. Since each $x_n = \inf\{y : F_{X_n}(y) = 1\}$, we must have $x - \epsilon \leq x_n \leq x + \epsilon$ for all $n \geq N$. Thus, $x_n \to x$.

(c) Note that for y not in the set of (countably many) discontinuities of Y, we have $F_{X_n}(y) \to F(y)$; since F_{X_n} only takes values in $\{0, 1\}$, this means that the limit function F_y must only take values in $\{0, 1\}$. The gaps at the point of discontinuity can be filled in by right continuity of F_Y; for any such point of discontinuity y, one can always find a sequence $\{y_k\}$ decreasing to y that misses all discontinuity points of F_Y, hence $F_Y(y_n) \to F_Y(y)$.

Set $x = \inf\{y : F_Y(y) = 1\}$. Then, $F_Y(y) = 0$ for all $y < x$, and $F_Y(y) = 1$ for all $y > x$ by construction, thus $P(Y = x) = 1$ (right continuity).

Exercise 2 Let X and Y be two real random variables. Show that $P(X = Y) = 1$ if and only if $E(f(X)) = E(f(Y))$ for all $f : \mathbb{R} \to \mathbb{R}$ which are bounded and uniformly continuous. Hence or otherwise, show that a sequence of probability measures cannot converge weakly to two different limits.

Solution. ($\Rightarrow$) This follows immediately from the fact that X and Y are identically distributed.

($\Leftarrow$) Let a, b be continuity points of both F_X, F_Y, with $-\infty < a < b < \infty$. Pick a sequence δ_k decreasing to 0, and define the functions

$$h_k(x) = \begin{cases} 0, & \text{if } x \in (-\infty, a - \delta_k], \\ (x - (a - \delta_k))/\delta_k, & \text{if } x \in [a - \delta_k, a], \\ 1, & \text{if } x \in [a, b], \\ ((b + \delta_k) - x)/\delta_k, & \text{if } x \in [b, b + \delta_k], \\ 0, & \text{if } x \in [b + \delta_k, \infty). \end{cases}$$

Such $h_k \in C_B(\mathbb{R})$ approximate $\chi_{(a,b]}$ from above. Now, each h_k is bounded and uniformly continuous, so

$$F_X(b) - F_X(a) = E(\chi_{(a,b]}(X)) \leq E(h_k(X)) = E(h_k(Y)).$$

By the Dominated Convergence Theorem, taking limits as $k \to \infty$ gives

$$F_X(b) - F_X(a) \leq E(\chi_{[a,b]}(Y)) = E(\chi_{(a,b]}(Y)) = F_Y(b) - F_Y(a).$$

The reverse inequality holds by symmetry, hence

$$F_X(b) - F_X(a) = F_Y(b) - F_Y(a).$$

Taking the limit $a \to -\infty$, we have $F_X(b) = F_Y(b)$ for all common continuity points b. Thus, X and Y are identically distributed.

With this, we see that if $X_n \xrightarrow{d} X, Y$, then for all $f \in C_B(\mathbb{R})$, we have $E(f(X_n)) \to E(f(X))$, $E(f(X_n)) \to E(f(Y))$. Thus, $E(f(X)) = E(f(Y))$ for all $f \in C_B(\mathbb{R})$, whence X and Y are identically distributed.

Exercise 3

(a) Show that for any real random variable X and any proper open subset U of $\mathbb{R}$, we have

$$P(X \in U) = \sup\{P(X \in K) : K \subseteq U \text{ is compact}\}.$$

(b) Show that

$$X_n \xrightarrow{d} X \iff \liminf P(X_n \in U) \geq P(X \in U) \text{ for any open } U \subseteq \mathbb{R}.$$

(c) Show that

$$X_n \xrightarrow{d} X \iff \limsup P(X_n \in F) \leq P(X \in F) \text{ for any closed } F \subseteq \mathbb{R}.$$

Solution.

(a) We use the property of inner regularity of probability measures,

$$P(X \in U) = \sup\{P(X \in F) : F \subseteq U \text{ is closed}\}.$$

Given $\epsilon > 0$, find $N > 0$ such that $P(X \in U \setminus [-N, N]) < \epsilon/2$; this is possible, since this probability converges to zero as $N \to \infty$ by continuity from above. Via inner regularity, find closed $F \subseteq U \cap [-N, N]$ such that $P(X \in U \cap [-N, N]) - P(X \in F) < \epsilon/2$. Note that $F \subseteq [-N, N]$, hence F is compact. Also,

$$\begin{aligned} P(X \in U) - P(X \in F) &\leq P(X \in U \setminus [-N, N]) + P(X \in U \cap [-N, N]) - P(X \in F) \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

(b) ($\Rightarrow$) Note that $E(f(X_n)) \to E(f(X))$ for all continuous and bounded functions f. Let $U \subseteq \mathbb{R}$ be open. For any compact set $K \subseteq U$, use Urysohn's Lemma to find $f \in C_c(\mathbb{R})$ such that $0 \leq f \leq 1$, $f(K) = 1$, and $\operatorname{supp}(f) \subseteq U$. Thus,

$$P(X_n \in U) = E(\chi_U(X_n)) \geq E(f(X_n)).$$

As $n \to \infty$,

$$\liminf_{n\to\infty} P(X_n \in U) \geq E(f(X)) \geq E(\chi_K(X)) = P(X \in K).$$

Taking a supremum over all such K, and using the previous exercise gives

$$\liminf_{n\to\infty} P(X_n \in U) \geq P(X \in U).$$

($\Leftarrow$) Let x be a continuity point of X. Considering the open sets $(-\infty, x)$ and (x, ∞), we have

$$\liminf_{n\to\infty} P(X_n < x) \geq P(X < x), \qquad \liminf_{n\to\infty} P(X_n > x) \geq P(X > x).$$

The latter gives

$$\limsup_{n\to\infty} P(X_n \leq x) = 1 - \liminf_{n\to\infty} P(X_n > x) \leq 1 - P(X > x) = P(X \leq x).$$

This gives

$$\begin{aligned}P(X \le x) = P(X < x) &\le \liminf_{n\to\infty} P(X_n < x) \\ &\le \liminf_{n\to\infty} P(X_n \le x) \\ &\le \limsup_{n\to\infty} P(X_n \le x) \\ &\le P(X \le x).\end{aligned}$$

Thus, $P(X_n \le x) \to P(X \le x)$, i.e. $X_n \xrightarrow{d} X$.

(c) The conditions

$$\liminf P(X_n \in U) \ge P(X \in U) \text{ for any open } U \subseteq \mathbb{R},$$
$$\limsup P(X_n \in F) \le P(X \in F) \text{ for any closed } F \subseteq \mathbb{R},$$

are equivalent, since

$$\liminf P(X_n \in U) = 1 - \limsup P(X_n \in U^c), \qquad \limsup P(X_n \in F) = 1 - \liminf P(X_n \in F^c),$$

and

$$P(X \in U) = 1 - P(X \in U^c), \qquad P(X \in F) = 1 - P(X \in F^c).$$

Exercise 4 Let $\mathcal{A}$ be a π-system of subsets of $\mathbb{R}$. Suppose that $\mathcal{A}$ has the property that any open subset of $\mathbb{R}$ is a countable union of sets from $\mathcal{A}$. For a collection of real random variables $\{X_n\}_{n\ge1}$ and X, show that $P(X_n \in A) \to P(X \in A)$ for every $A \in \mathcal{A}$ implies that $X_n \xrightarrow{d} X$.

Solution. We use the criterion from 3(b); let $U \subseteq \mathbb{R}$ be open. Then, write

$$U = \bigcup_{n=1}^{\infty} A_n, \qquad U_N = \bigcup_{n=1}^{N} A_n$$

where $A_n \in \mathcal{A}$. Then, the Inclusion-Exclusion Principle gives

$$P(X_n \in U_N) = \sum_{\emptyset \ne J \subseteq \{1,\dots,N\}} (-1)^{|J|+1} P\left(X_n \in \bigcap_{j\in J} A_j\right).$$

Taking the limit $n \to \infty$, we have

$$\lim_{n\to\infty} P(X_n \in U_N) = \sum_{\emptyset \ne J \subseteq \{1,\dots,N\}} (-1)^{|J|+1} P\left(X \in \bigcap_{j\in J} A_j\right) = P(X \in U_N).$$

Since each $P(X_n \in U) \ge P(X_n \in U_N)$,

$$\liminf_{n\to\infty} P(X_n \in U) \ge P(X \in U_N),$$

so taking the limit $N \to \infty$ and using continuity from below,

$$\liminf_{n\to\infty} P(X_n \in U) \ge P(X \in U).$$

Exercise 5 Let $\{X_n\}_{n\ge1}$ be i.i.d. real random variables defined from a probability space $(\Omega, \mathcal{F}, P)$ having a common cdf F. Define the empirical cdf $\hat{F}_n$ corresponding to the observations $(X_1, \dots, X_n)$ as

$$\hat{F}_n(\omega, x) = \frac{1}{n}\sum_{i=1}^{n} \chi_{(X_i(\omega)\le x)},$$

where $\omega \in \Omega$ and $x \in \mathbb{R}$.

(a) Prove that $\hat{F}_n(\cdot, x)$ is a real-valued random variable for each $x \in \mathbb{R}$.

(b) Prove that $\hat{F}_n(\omega, \cdot)$ is a valid cdf for each $\omega \in \Omega$.

(c) Show that $\hat{F}_n(\cdot, \cdot) \xrightarrow{d} F$ as $n \to \infty$ almost surely, i.e.

$$P(\{\omega : \lim_{n\to\infty} \hat{F}_n(\omega, x) = F(x) \text{ for every } x \in C(F)\}) = 1.$$

Solution.

(a) Note that each of the terms $Y_i = \chi_{[X_i \le x]}$ is a real-valued random variable; since $X_i : \Omega \to \mathbb{R}$ is measurable, the set $[X_i \le x] = X_i^{-1}(-\infty, x]$ is measurable, whence the indicator function Y_i on this set is also measurable. Thus, $\hat{F}_n(\cdot, x) : \Omega \to \mathbb{R}$ is measurable, being a scaled finite sum of measurable functions. This means that $\hat{F}_n(\cdot, x)$ is a random variable for each $x \in \mathbb{R}$.

(b) It is enough to check that given $\omega \in \Omega$, the map $\hat{F}_n(\omega, \cdot)$ is non-decreasing, right continuous, $\lim_{x\to-\infty} \hat{F}_n(\omega, x) = 0$, and $\lim_{x\to\infty} \hat{F}_n(\omega, x) = 1$.

Denote $X_i(\omega) = y_j$, where the indices j are arranged such that $y_1 \le \cdots \le y_n$. Then,

$$\hat{F}(\omega, x) = \begin{cases} 0 & \text{if } x \in (-\infty, y_1), \\ 1/n & \text{if } x \in [y_1, y_2), \\ \vdots & \vdots \\ k/n & \text{if } x \in [y_k, y_{k+1}), \\ \vdots & \vdots \\ 1 & \text{if } x \in [y_n, \infty). \end{cases}$$

All four properties follow immediately.

(c) For fixed x, the random variable $Y_i = \chi_{[X_i \le x]}$ has expectation $F(x)$, which is also the absolute expectation. Since all X_i are independent, so are all Y_i, whence the Strong Law of Large Numbers gives

$$\hat{F}_n(\cdot, x) = \frac{1}{n} \sum_{i=1}^{n} Y_i \xrightarrow{as} E(Y_1) = F(x).$$

Note that the (countable) collection $\mathcal{A}$ of all intervals of the form (q_i, q_i') for $q_i, q_i' \in \mathbb{Q}$ forms a basis of the standard topology on $\mathbb{R}$. Now, for each $x \in C(F)$,

$$P(\hat{F}_n(\cdot, x) < q_i') \to P(F(x) < q_i'), \qquad P(\hat{F}_n(\cdot, x) \le q_i) \to P(F(x) \le q_i),$$

since $\hat{F}_n(\cdot, x) \xrightarrow{as} F(x)$ implies $\hat{F}_n(\cdot, x) \xrightarrow{d} F(x)$. Thus,

$$P(\hat{F}_n(\cdot, x) \in A) \to P(F(x) \in A)$$

for every $A \in \mathcal{A}$, whence $\mathcal{A}$ is a π-system as described in the previous exercise. Thus, $\hat{F}(\cdot, x) \xrightarrow{d} F(x)$ for all $x \in C(F)$, hence $\hat{F}(\cdot, \cdot) \xrightarrow{d} F$ almost surely.

Assignment III

Exercise 1 Let X be a random variable with characteristic function ϕ_X. Show that the following are equivalent.

(a) $|\phi_X(t_0)| = 1$ for some $t_0 \neq 0$.

(b) There exists $a, h \in \mathbb{R}$ with $h \neq 0$ such that

$$P(X \in \{a + kh : k \in \mathbb{Z}\}) = 1.$$

Solution. ($\Leftarrow$) Check that

$$\phi_X(2\pi/h) = E(e^{2\pi i X/h}) = \sum_{k\in\mathbb{Z}} e^{2\pi i(a+kh)/h} P(X = a + kh) = e^{2\pi i a/h} \sum_{k\in\mathbb{Z}} P(X = a + kh) = e^{2\pi i a/h}.$$

($\Rightarrow$) Let $|\phi_X(t_0)| = 1$ for some $t_0 \neq 0$. Then, write $\phi_X(t_0) = e^{2\pi i\alpha}$, and set $h = 2\pi/t_0$, $a = \alpha h = 2\pi\alpha/t_0$. Now,

$$\begin{aligned}
1 &= e^{-2\pi i\alpha}\phi_X(t_0) \\
&= e^{-2\pi i\alpha} \int_{\mathbb{R}} e^{it_0 x}\, dF(x) \\
&= e^{-2\pi i a/h} \int_{\mathbb{R}} e^{2\pi i x/h}\, dF(x) \\
&= \int_{\mathbb{R}} e^{2\pi i(x-a)/h}\, dF(x),
\end{aligned}$$

so

$$\int_{\mathbb{R}} 1 - e^{2\pi i(x-a)/h}\, dF(x) = 0.$$

This is possible only when $e^{2\pi i(x-a)/h} = 1$ almost everywhere with respect to F, i.e. when $(X - a)/h \in \mathbb{Z}$ almost surely.

To see this, note that the above equation forces

$$\int_{\mathbb{R}} 1 - \cos(2\pi(x-a)/h)\, dF(x) = 0,$$

hence

$$\int_{\mathbb{R}} \sin^2(\pi(x-a)/h)\, dF(x) = 0.$$

Exercise 2 Let F be a cdf on $\mathbb{R}$ with pdf f and c.f. ϕ_X. Show that

$$\lim_{|t|\to\infty} |\phi_X(t)| = 0.$$

Solution. First, suppose that $f \in C_c(\mathbb{R})$. Then, note that the substitution $x \mapsto x + \pi/t$ gives

$$\phi_X(t) = \frac{1}{2\pi} \int_{\mathbb{R}} f(x)e^{itx}\, dx = \frac{1}{2\pi} \int_{\mathbb{R}} f(x + \pi/t)e^{itx}e^{i\pi}\, dx.$$

Taking averages,

$$\phi_X(t) = \frac{1}{2\pi} \int_{\mathbb{R}} [f(x) - f(x + \pi/t)]e^{itx}\, dt,$$

so

$$|\phi_X(t)| \leq \frac{1}{2\pi} \int_{\mathbb{R}} |f(x) - f(x + \pi/t)|\, dx.$$

Note that as $|t| \to \infty$, we have $|f(x) - f(x + \pi/t)| \to 0$; also, for $|t| > 1$, the latter has compact hence bounded support. Thus, the Bounded Convergence Theorem gives

$$\lim_{|t|\to\infty} |\phi_X(t)| = 0.$$

Next, if $f \notin C_c(\mathbb{R})$, use the density of $C_c(\mathbb{R})$ in $L^1(\mathbb{R})$ to find $g \in C_c(\mathbb{R})$ such that $\|f - g\|_1 < \epsilon$. Then, separate

$$\phi_X(t) = \frac{1}{2\pi} \int_{\mathbb{R}} [f(x) - g(x)] e^{itx}\, dx + \frac{1}{2\pi} \int_{\mathbb{R}} g(x) e^{itx}\, dx,$$

hence

$$|\phi_X(t)| \leq \frac{1}{2\pi} \|f(x) - g(x)\|_1 + \frac{1}{2\pi} |\int_{\mathbb{R}} g(x) e^{itx}\, dx|.$$

The first term is less than $\epsilon/2\pi$, and the second vanishes in the limit $|t| \to \infty$ by the previous argument. Since $\epsilon > 0$ is arbitrary, we have

$$\lim_{|t|\to\infty} |\phi_X(t)| = 0.$$

Exercise 3 Let X and Y be i.i.d. random variables with cdf F and c.f. ϕ. Show that

$$P(X = Y) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\phi(t)|^2\, dt.$$

Hence or otherwise, show that F is continuous if and only if the above limit equals zero.

Solution. Set $Z = X - Y$, and note that the independence of X, Y, followed by their identical distribution gives

$$\phi_Z(t) = \phi_{X-Y}(t) = \phi_X(t)\phi_{-Y}(t) = \phi(t)\,\overline{\phi(t)} = |\phi(t)|^2.$$

Thus, the inversion formulae give

$$P(X = Y) = P(Z = 0) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \phi_Z(t)\, dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\phi(t)|^2\, dt.$$

We show that F is continuous if and only if $P(X = Y) = 0$. To see this, note that $P(X = Y) = P((X, Y) \in \Delta)$, where $\Delta = \{(x, x) : x \in \mathbb{R}\}$. Thus,

$$P(X = Y) = \iint_{\Delta} dF_{X,Y}(x, y) = \int_{\mathbb{R}} \int_{\{x=y\}} dF(x)\, dF(y).$$

If F is continuous, the inner integral is always zero, hence $P(X = Y) = 0$. Conversely, if $P(X = a) = p > 0$ for some $a \in \mathbb{R}$, then $P(X = Y) \geq P((X, Y) = (a, a)) = p^2 > 0$.

Exercise 4 Let $\{F_n\}$ and F be cdfs on $\mathbb{R}$ with corresponding c.f.'s given by $\{\phi_n\}$ and ϕ. Suppose that $F_n \xrightarrow{d} F$.

(a) Give an example to show that ϕ_n may not converge to ϕ uniformly on the entire real line.

(b) Suppose that F_n and F have pdfs given by f_n and f. If f_n converges to f almost surely, then show that ϕ_n converges to ϕ uniformly on $\mathbb{R}$.

Solution.

(a) Let $F_n \sim N(0, 1/n)$, whence $\phi_n(t) = e^{-t^2/2n}$. As $n \to \infty$, we have $\phi_n \to 1$, hence $F_n \xrightarrow{d} F$ where F is the cdf of a degenerate distribution with full mass at 0. However, ϕ_n does not converge uniformly to 1 on $\mathbb{R}$. Note that $\phi_n(\sqrt{2n}) = e^{-1}$, hence

$$\|\phi_n - 1\| = \sup_{t\in\mathbb{R}} |\phi_n(t) - 1| \geq |e^{-1} - 1| \not\to 0.$$

(b) Write

$$\phi_n(t) - \phi(t) = \frac{1}{2\pi} \int_{\mathbb{R}} (f_n(x) - f(x)) e^{itx} \, dx,$$

hence

$$|\phi_n(t) - \phi(t)| \leq \frac{1}{2\pi} \int_{\mathbb{R}} |f_n(x) - f(x)| \, dx.$$

As $n \to \infty$, the right hand side (which is independent of t) converges to zero by the Dominated Convergence Theorem, since $f_n \to f$ almost surely, and

$$\int_{\mathbb{R}} |f_n(x) - f(x)| \, dx = 2 \int_{\mathbb{R}} (f(x) - f_n(x))^+ \, dx,$$

with $(f - f_n)^+$ dominated by f which is integrable on $\mathbb{R}$.

Exercise 5 Let ϕ_X be the c.f. of a random variable X on $\mathbb{R}$. Suppose that $|\phi_X(t)| = |\phi_X(\alpha t)| = 1$ for some non-zero $t \in \mathbb{R}$, and some irrational $\alpha \in \mathbb{R}$. Show that there exists $c \in \mathbb{R}$ such that $P(X = c) = 1$.

Solution. From Exercise 1, we find that X must be supported on

$$S = \{a + kh : k \in \mathbb{Z}\} \cap \{a/\alpha + kh/\alpha : k \in \mathbb{Z}\},$$

where $h = 2\pi/t$, $a = \beta h$, $\phi_X(t) = e^{2\pi i \beta}$. Any element $x \in S$ must look like

$$x = (\beta + k)h = (\beta + \ell)h/\alpha$$

for $k, \ell \in \mathbb{Z}$. Thus,

$$\alpha = \frac{\beta + \ell}{\beta + k}.$$

If we had $x' \in S$ with $x' \neq x$, then we could write

$$\alpha = \frac{\beta + \ell'}{\beta + k'}$$

for $k', \ell' \in \mathbb{Z}$, $k' \neq k$, $\ell' \neq \ell$. Thus,

$$\alpha = \frac{\beta + \ell}{\beta + k} = \frac{\beta + \ell'}{\beta + k'} = \frac{(\beta + \ell) - (\beta + \ell')}{(\beta + k) - (\beta + k')} = \frac{\ell - \ell'}{k - k'},$$

contradicting the irrationality of α. Thus, S contains at most one element; it must contain at least one element since $P(X \in S) = 1$.

Exercise 6 Let $\{X_n\}$ and X be a collection of random variables with corresponding c.f.'s $\{\phi_n\}$ and ϕ. Suppose that $\phi_n \in L^1(\mathbb{R})$ for each $n \geq 1$, and ϕ_n converges in $L^1(\mathbb{R})$ to ϕ. Show that

$$\sup_{B \in \mathcal{B}_{\mathbb{R}}} |P(X_n \in B) - P(X \in B)| \to 0.$$

Solution. Note that $\phi \in L^1(\mathbb{R})$; thus, $\{X_n\}$ and X admit density functions f_n and f. By Scheffe's Theorem, it is now enough to show that $f_n \to f$ almost everywhere. To do so, use the inversion formula

$$f_n(x) - f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} (\phi_n(t) - \phi(t)) e^{-itx} \, dt,$$

hence

$$|f_n(x) - f(x)| \leq \frac{1}{2\pi} \|\phi_n - \phi\|_1 \to 0.$$

Exercise 7 Let $\{U_i\}_{i\geq 1}$ be i.i.d. random variables with distribution $P(U_1 = \pm 1) = 1/2$. Define $X_n = \sum_{i=1}^n U_i/2^i$.

(a) Find the c.f. of X_n.

(b) Show that $\lim_{n\to\infty} X_n$ exists almost surely, and denote it by X. Show that the c.f. of X is given by $\phi_X(t) = \sin(t)/t$.

Solution.

(a) Calculate

$$\phi_{U_1}(t) = E(e^{itU_1}) = \frac{1}{2}e^{it} + \frac{1}{2}e^{-it} = \cos(t).$$

Thus, using independence,

$$\phi_{X_n}(t) = \prod_{i=1}^n \phi_{U_i/2^i}(t) = \prod_{i=1}^n \phi_{U_1}(t/2^i) = \prod_{i=1}^n \cos(t/2^i).$$

Multiplying and dividing by $\sin(t/2^i)$ and using the identity $2\sin(x)\cos(x) = \sin(2x)$, we can simplify this (for $t \neq 0$) to

$$\phi_{X_n}(t) = \frac{\sin(t)}{2^n \sin(t/2^n)}.$$

(b) Note that

$$\lim_{n\to\infty} X_n = \lim_{n\to\infty} \sum_{i=1}^n U_i/2^i$$

is an infinite sum of centred random variables; it is enough to check that the following limit is finite.

$$\lim_{n\to\infty} \sum_{i=1}^n V(U_i/2^i) = \lim_{n\to\infty} \sum_{i=1}^n \frac{1/4}{2^{2i}} = \frac{1/4}{1-1/4} = \frac{1}{3}.$$

Thus, X_n converges almost surely, say $X_n \xrightarrow{as} X$. This means that $X_n \xrightarrow{d} X$, hence $\phi_{X_n} \to \phi_X$. Calculate

$$\lim_{n\to\infty} \phi_{X_n}(t) = \lim_{n\to\infty} \frac{\sin(t)}{2^n \sin(t/2^n)} = \lim_{n\to\infty} \frac{\sin(t)}{t} \cdot \frac{t/2^n}{\sin(t/2^n)} = \frac{\sin(t)}{t}.$$

This is precisely the c.f. of a $U(-1,1)$ random variable.

Exercise 8 Let $\{X_n\}$ be a sequence of independent random variables with $P(X = \pm 1) = 1/2 - 1/2\sqrt{n}$ and $P(X = \pm n^2) = 1/2\sqrt{n}$ for each $n \geq 1$. Find constants $\{a_n\} \subseteq (0,\infty)$ and $\{b_n\} \subseteq \mathbb{R}$ such that $a_n^{-1}\sum_{j=1}^n (X_n - b_n)$ converges weakly to $N(0,1)$.

Solution. Note that $E(X_j) = 0$; set

$$\sigma_j^2 = V(X_j) = 1 - \frac{1}{\sqrt{n}} + \frac{n^4}{\sqrt{n}}, \qquad s_n^2 = \sum_{j=1}^n \sigma_j^2.$$

We claim that $a_n = s_n$, $b_n = 0$ gives the desired result, via the Lindeberg-Levy Central Limit Theorem. Check that σ_j^2 increases to ∞, hence

$$\frac{\max_{1\leq j\leq n} \sigma_j^2}{s_n^2} = \frac{\sigma_n^2}{s_n^2}.$$

Now,

$$n^{7/2} \leq \sigma_n^2 \leq 1 + n^{7/2},$$

hence

$$\sum_{j=1}^n j^{7/2} \leq s_n^2 \leq n + \sum_{j=1}^n j^{7/2}$$

Also,

$$\int_{j-1}^{j} x^{7/2}\,dx \le j^{7/2} \le \int_{j}^{j+1} x^{7/2}\,dx,$$

so

$$\frac{2}{9}n^{9/2} \le \int_0^n x^{7/2}\,dx \le s_n^2 \le \int_1^{n+1} x^{7/2}\,dx = \frac{2}{9}(n+1)^{9/2}.$$

Thus,

$$\frac{\max_{1\le j\le n}\sigma_j^2}{s_n^2} = \frac{\sigma_n^2}{s_n^2} \le \frac{1+n^{7/2}}{2n^{9/2}/9} \to 0.$$

Next, we verify the Lyapunov condition for $\delta = 2$. Check that

$$E(X_j^4) = 1 - \frac{1}{\sqrt{n}} + \frac{n^8}{\sqrt{n}} \le 1 + n^{15/2},$$

hence

$$\sum_{j=1}^{n} E(X_j^4) \le n + \sum_{j=1}^{n} n^{15/2} \le n + \int_1^{n+1} n^{15/2} = n + \frac{2}{17}n^{17/2}.$$

Thus,

$$\frac{\sum_{j=1}^n E(X_j^4)}{s_n^4} \le \frac{n + 2n^{17/2}/17}{4n^9/81} \to 0.$$

Lindeberg-Levy now gives

$$s_n^{-1}\sum_{j=1}^{n} X_j \xrightarrow{d} N(0,1).$$

Exercise 9 Let $\{X_n\}$ be a sequence of random variables. Let

$$S_n = \sum_{j=1}^{n} X_j, \qquad s_n^2 = \sum_{j=1}^{n} E(X_j^2) < \infty.$$

If $s_n^2 \to \infty$, then show that the following are equivalent.

$$\lim_{n\to\infty} s_n^{-2}\sum_{j=1}^{n} E(X_j^2\chi_{|X_j|>\epsilon s_n}) = 0 \text{ for all } \epsilon > 0,$$

$$\lim_{n\to\infty} s_n^{-2}\sum_{j=1}^{n} E(X_j^2\chi_{|X_j|>\epsilon s_j}) = 0 \text{ for all } \epsilon > 0,$$

Solution. ($\Leftarrow$) Each sum in the second expression has more terms than in the first, since $|X_j| > \epsilon s_n \implies |X_j| > \epsilon s_j$. Thus, the first expression is sandwiched between zero and the second expression, hence must also be zero in the limit.

($\Rightarrow$) Check that for any $\delta > 0$, we have

$$\sum_{j:s_j<\delta s_n} E(X_j^2) < \delta^2 s_n^2.$$

This is clear, since s_j increases to s_n; if $j'(n)$ is the largest j such that $s_j < \delta s_n$, then the sum is over precisely $1, 2, \ldots, j'(n)$, hence is equal to $s_{j'(n)}^2$. But $s_{j'(n)}^2 < \delta^2 s_n^2$ by construction.

Let $\epsilon > 0$; for each $\delta > 0$, we have

$$s_n^{-2}\sum_{j=1}^{n} E(X_j^2\chi_{|X_j|>\delta\epsilon s_n}) \to 0.$$

Now,

$$\begin{aligned}
s_n^{-2}\sum_{j=1}^{n} E(X_j^2\chi_{|X_j|>\epsilon s_j}) &= s_n^{-2}\sum_{j=1}^{n} E(X_j^2\chi_{|X_j|>\epsilon s_j}(\chi_{|X_j|>\delta\epsilon s_n}+\chi_{|X_j|\le\delta\epsilon s_n})) \\
&= s_n^{-2}\sum_{j=1}^{n} E(X_j^2\chi_{|X_j|>\epsilon s_j}\chi_{|X_j|>\delta\epsilon s_n}) + s_n^{-2}\sum_{j=1}^{n} E(X_j^2\chi_{|X_j|>\epsilon s_j}\chi_{|X_j|\le\delta\epsilon s_n}) \\
&\le s_n^{-2}\sum_{j=1}^{n} E(X_j^2\chi_{|X_j|>\delta\epsilon s_n}) + s_n^{-2}\sum_{j=1}^{n} E(X_j^2\chi_{\delta\epsilon s_n\ge|X_j|>\epsilon s_j}) \\
&\le s_n^{-2}\sum_{j=1}^{n} E(X_j^2\chi_{|X_j|>\delta\epsilon s_n}) + s_n^{-2}\sum_{j=1}^{n} E(X_j^2\chi_{\delta s_n>s_j}) \\
&< s_n^{-2}\sum_{j=1}^{n} E(X_j^2\chi_{|X_j|>\delta\epsilon s_n}) + \delta^2
\end{aligned}$$

Taking the limit as $n\to\infty$, the first term vanishes. Since $\delta>0$ is arbitrary, the limit must be zero.