MA4103

Analysis V

Autumn 2022

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Contents

1 Signed measures

As a motivating example, consider the space of linear functionals on $\mathbb{C}^n$; each element of this space is a linear function $\phi\colon \mathbb{C}^n \to \mathbb{C}$. If we denote the standard basis of $\mathbb{C}^n$ by $\{e_i\}$ and its dual basis by $\{\phi_i\}$, any linear functional ϕ can be written as

$$\phi = \sum_{i=1}^{n} \phi(e_i)\,\phi_i.$$

Now we switch to a different perspective; note that each element of $\mathbb{C}^n$ is an n-tuple of complex numbers of the form $(z_1, z_n, \ldots, z_n)$. In other words, it is an assignment of complex numbers to the indices $X = \{1, 2, \ldots, n\}$. Thus, each element of $\mathbb{C}^n$ can be associated with a function $f\colon X \to \mathbb{C}$, and vice versa. This gives us an identification of $\mathbb{C}^n$ with the space of functions $C(X)$ (all such functions are continuous when X is equipped with the discrete topology). For instance, the basis vectors $\{e_i\}$ of $C(X)$ can now be represented by the functions

$$e_i\colon X \to \mathbb{C}, \qquad j \mapsto \delta_{ij}.$$

Now, in order to extract the 'coordinates' $z_i \equiv f(i)$ from some $f \in C(X)$, we may define the following Dirac measures μ_i on X, concentrated at the index i.

$$\mu_i\colon \mathcal{P}(X) \to C, \qquad E \mapsto \begin{cases} 1, & \text{if } i \in A, \\ 0, & \text{otherwise.} \end{cases}$$

This means that for any $f \in C(X)$,

$$f(i) = \int f\, d\mu_i.$$

The dual basis $\{\phi_i\}$ behaves as $\phi_i(e_j) = \delta_{ij}$, so

$$\phi_i(f) = \phi_i \sum_{j=1}^{n} f(j)\, e_j = f(i) = \int f\, d\mu_i.$$

Thus, for any linear functional $\phi\colon C(X) \to \mathbb{C}$, we have

$$\phi(f) = \sum_{i=1}^{n} \phi(e_i)\,\phi_i(f) = \sum_{i=1}^{n} \phi(e_i) \int f\, d\mu_i = \int f \sum_{i=1}^{n} \phi(e_i)\, d\mu_i.$$

If we could make sense of the measure μ described by

$$\mu = \sum_{i=1}^{n} \phi(e_i)\,\mu_i,$$

then we could write

$$\phi(f) = \int f\, d\mu.$$

Note that the coefficients $\phi(e_i)$ are complex numbers, so μ is not necessarily a measure in the conventional sense!

1.1 Basic definitions

Definition 1.1. Let $\mathcal{M}$ be a σ-algebra over X. A function $\nu\colon \mathcal{M} \to [-\infty, +\infty]$ is called a signed measure if

1. $\nu(\emptyset) = 0$.
2. For all countable collections of disjoint measurable sets $\{E_i\}$, we have

$$\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \nu(E_i).$$

Remark. For disjoint sets $E, F \in \mathcal{M}$, we want

$$\nu(E \cup F) = \nu(E) + \nu(F).$$

Thus, we cannot allow the situation where $\nu(E) = +\infty$ and $\nu(F) = -\infty$, or vice versa.

Remark. Note that in condition 2, the union on the left hand side is independent of order, while the infinite sum on the right hand side is not. Thus, we demand that either the sum on the right converges absolutely, or the following: neither one of the sums

$$\sum_{i:\nu(E_i)\geq 0} \nu(E_i), \qquad \sum_{j:\nu(E_j)<0} \nu(E_j)$$

should diverge to ∞.

Lemma 1.1. *Let $E, F \in \mathcal{M}$, with $E \subseteq F$. Then,*

1. *If $|\nu(E)| < \infty$, then*
$$\nu(F \setminus E) = \nu(F) - \nu(E).$$
2. *If $\nu(E) = \pm\infty$, then $\nu(F) = \pm\infty$.*

Proof. Using $\nu(F) = \nu(F \setminus E) + \nu(E)$, we obtain *1* when $|\nu(E)| < \infty$. Otherwise, for $\nu(E) = \infty$, we cannot have $\nu(F \setminus E) = \mp\infty$, hence we must have $\nu(F) = \pm\infty$. □

Corollary 1.1.1. *If any measurable set $E \subseteq X$ has $\nu(E) = \pm\infty$, then $\nu(X) = \pm\infty$. This immediately shows that we cannot have two measurable sets $E, F \in \mathcal{M}$ with $\nu(E) = +\infty$, $\nu(F) = -\infty$. In other words, a signed measure has either of the following forms.*

$$\nu\colon \mathcal{M} \to [-\infty, +\infty), \quad \text{or} \quad \nu\colon \mathcal{M} \to (-\infty, +\infty].$$

Example. Consider a non-negative measure μ, and a function $f \in L^1(\mu)$. Then, the measure defined by

$$\nu(E) = \int_E f \, d\mu$$

is a (finite) signed measure.

Example. Consider any two non-negative measures $\mu_1, \mu_2 \geq 0$. Then, the measure ν defined by

$$\nu = \mu_1 - \mu_2$$

where either one of μ_1, μ_2 is finite is a signed measure. Indeed, any signed measure is of this form.

Lemma 1.2 (Continuity from below)**.** *Let $\{E_i\}$ be a collection of measurable sets such that each $E_i \subseteq E_{i+1}$. Then,*

$$\lim_{n\to\infty} \nu(E_n) = \nu\left(\bigcup_{n=1}^{\infty} E_n\right).$$

Lemma 1.3 (Continuity from above)**.** *Let $\{E_i\}$ be a collection of measurable sets such that each $E_i \supseteq E_{i+1}$, and $|\nu(E_1)| < \infty$. Then,*

$$\lim_{n\to\infty} \nu(E_n) = \nu\left(\bigcap_{n=1}^{\infty} E_n\right).$$

1.2 The Hahn-Jordan decomposition theorems

Definition 1.2. A set $P \in \mathcal{M}$ is called a positive set for ν if for every measurable subset $E \subseteq P$, we have $\nu(E) \geq 0$.

Definition 1.3. A set $N \in \mathcal{M}$ is called a negative set for ν if for every measurable subset $E \subseteq N$, we have $\nu(E) \leq 0$.

Definition 1.4. A set $F \in \mathcal{M}$ is called a null set for ν if for every measurable subset $E \subseteq F$, we have $\nu(E) = 0$.

Example. Let μ be a non-negative measure and let f be a measurable function. Define the signed measure

$$\nu(E) = \int_E f \, d\mu.$$

It is clear that any measurable subset of $f^{-1}[0, \infty]$ is a positive set for ν, any measurable subset of $f^{-1}[-\infty, 0]$ is a negative set for ν, and any measurable subset of $f^{-1}(0)$ is a null set for ν.

Lemma 1.4. *Every measurable subset of a positive(/negative/null) set is positive(/negative/null).*

Proof. Let P be a positive set for ν, and $E \subseteq P$ be a measurable subset. We claim that for all measurable subsets $F \subseteq E$, $\nu(F) \geq 0$. This is immediate from the fact that $F \subseteq E \subseteq P$ is a measurable subset of the positive set P. The cases for negative and null sets are analogous. □

Lemma 1.5. *Countable unions of positive(/negative/null) sets are positive(/negative/null).*

Proof. Let $\{P_i\}$ be a countable collection of positive measurable sets, and let $P = \bigcup_{i=1}^{\infty} P_i$. Define the sets

$$Q_i = P_i \setminus \bigcup_{j=1}^{i-1} P_j,$$

and note that $P = \bigcup_{i=1}^{\infty} Q_i$, with the collection $\{Q_i\}$ being disjoint. Furthermore, each $Q_i \subseteq P_i$ is a positive set. Now for any measurable $E \subseteq P$, each $E \cap Q_i \subseteq Q_i$ is a positive set, hence

$$E = \bigcup_{i=1}^{\infty} E \cap Q_i = \sum_{i=1}^{\infty} \nu(E \cap Q_i) \geq 0.$$

The cases for negative and null sets are analogous. □

Lemma 1.6. *Let $\nu\colon \mathcal{M} \to [-\infty, +\infty)$ be a signed measure, and let $E \in \mathcal{M}$ such that $\nu(E) > 0$. Then, there exists a measurable subset $\tilde{E} \subseteq E$ such that $\tilde{E}$ is a positive set for ν and $\nu(\tilde{E}) > 0$.*

Proof. If E is a positive set, we are done. Otherwise, there exists some $F \subseteq E$ such that $\nu(F) < 0$. Note that

$$\nu(E \setminus F) = \nu(E) - \nu(F) > 0.$$

For any non-positive set A, define the set

$$S_A = \{n \in \mathbb{N} : \nu(B) < -1/n \text{ for some } B \subseteq A, B \in \mathcal{M}\}.$$

When A is not a positive set, S_A is clearly non-empty and hence has a minimum element by the Well Ordering Principle.

We have a non-positive set E, hence S_E has a minimum element n_1. Choose $F_1 \subseteq E$ such that $\nu(F_1) < -1/n_1$, set $E_1 = E \setminus F_1$, and note that

$$\nu(E_1) = \nu(E) - \nu(F_1) > \nu(E) + \frac{1}{n_1} > 0.$$

If E_1 is a positive set we are done. Otherwise, S_{E_1} has a minimum element n_2, hence we can pick $F_2 \subseteq F_1$ such that $\nu(F_2) < -1/n_2$. Setting $E_2 = E_1 \setminus F_2 = E \setminus (F_1 \cup F_2)$, we have

$$\nu(E_2) = \nu(E_1) - \nu(F_2) > \nu(E) + \frac{1}{n_1} + \frac{1}{n_2} > 0.$$

Again, if E_2 is a positive set, we are done; otherwise, we refine it to obtain E_3 as before, and so on. In this manner, if at any stage the set $E_k = E \setminus \bigcup_{i=1}^k F_i$ is not a positive set, we let n_{k+1} be the minimum element of S_{E_k}, pick $F_{k+1} \subseteq E_k$ such that $\nu(F_{k+1}) < -1/n_{k+1}$, and note that

$$\nu(E_{k+1}) = \nu(E_k) - \nu(F_{k+1}) > \nu(E) + \sum_{i=1}^{k+1} \frac{1}{n_i} > 0.$$

We stop this process if at any stage E_k is a positive set; otherwise, we obtain infinite sequences of sets $\{E_i\}$ and $\{F_i\}$. Set

$$A = E \setminus \bigcup_{i=1}^{\infty} F_i = \bigcap_{i=1}^{\infty} E_i.$$

We claim that A is a positive set for ν. Note that the sets $\{F_i\}$ are all disjoint subsets of E, which has finite measure, hence

$$\nu\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} \nu(F_i) > -\infty.$$

This shows that the series

$$\sum_{i=1}^{\infty} \frac{1}{n_i} < -\sum_{i=1}^{\infty} \nu(F_i) < \infty.$$

For this convergence to hold, the terms $1/n_i \to 0$, $n_i \to \infty$. Also note that

$$\nu(A) = \nu(E) - \sum_{i=1}^{\infty} \nu(F_i) > \nu(E) + \sum_{i=1}^{\infty} \frac{1}{n_i} > 0.$$

Suppose that A is not positive for ν. Then, S_A has a minimum element m, hence we can pick $B \subseteq A$ such that $\nu(B) < -1/m$. Since $n_i \to \infty$, we can fix k such that $n_k > m$. But,

$$B \subseteq A = \bigcap_{i=1}^{\infty} E_i \subseteq E_k, \qquad \nu(B) < -\frac{1}{m} < -\frac{1}{n_k}.$$

This contradicts the minimality of n_k with respect to the set S_{E_k}. This shows that $A \subseteq E$ is indeed a positive set of positive measure, as desired. □

Theorem 1.7 (Hahn Decomposition)**.** *Let ν be a signed measure on $(X, \mathcal{M})$. Then, there exists a positive set P and a negative set N for ν such that $P \cup N = X$, $P \cap N = \emptyset$. Furthermore, if P, N and P', N' are two decompositions of X, then the symmetric difference $P \Delta P' = N \Delta N'$ is a null set for ν.*

Proof. Without loss of generality, suppose that ν is of the form $\nu \colon \mathcal{M} \to [-\infty, +\infty)$. Let $\mathscr{P} \subseteq \mathcal{M}$ be the collection of all positive measurable sets for ν. Note that this collection is non-empty, since $\emptyset \in \mathscr{P}$. Set

$$m = \sup_{E \in \mathscr{P}} \nu(E).$$

Note that from the properties of the supremum, there exists a sequence of sets $\{P_i\}_{i \in \mathbb{N}}$ from $\mathscr{P}$ such that $\nu(P_i) \to m$. Set $P = \bigcup_{i=1}^{\infty} P_i$, and note that P is a positive set for ν, i.e. $P \in \mathscr{P}$. We claim that $\nu(P) = m$. To see this, define

$$Q_i = \bigcup_{j=1}^{i} P_i,$$

and note that $\{Q_i\}$ is an increasing sequence of positive measurable sets for ν. Also, each $Q_i \supseteq P_i$, so $0 \leq \nu(P_i) \leq \nu(Q_i) \leq m$. Taking limits, we have $\nu(Q_i) \to m$. Thus, continuity from below gives

$$\nu(P) = \nu\left(\bigcup_{i=1}^{\infty} Q_i\right) = \lim_{n\to\infty} \nu(Q_i) = m.$$

Now, set $N = X \setminus P$. We claim that N is a negative set for ν. Indeed, if not, then we could find some measurable $E \subseteq N$ with $\nu(E) > 0$. By the previous lemma, this yields a positive set $\tilde{E} \subseteq E$ for ν with $\nu(\tilde{E}) > 0$. Now, $\tilde{E} \cup P$ is a positive set for ν, i.e. $\tilde{E} \cup P \in \mathscr{P}$. Also, $\tilde{E} \cap P = \emptyset$, so

$$\nu(\tilde{E} \cup P) = \nu(\tilde{E}) + \nu(P) > m.$$

This contradicts the maximality of m.

Thus, we have obtained a Hahn decomposition P, N of X. If P', N' is another Hahn decomposition of X, then

$$P\Delta P = (P \setminus P') \cup (P' \setminus P) = (N^c \setminus N'^c) \cup (N'^c \setminus N^c) = (N' \setminus N) \cup (N \setminus N') = N\Delta N'.$$

Furthermore, $P \setminus P' \subseteq P$ is a positive set for ν; $P \setminus P' \subseteq P'^c = N'$ is also a negative set for ν. This means that $P \setminus P'$ must be a null set for ν. The same reasoning shows that $P' \setminus P$ is a null set for ν, hence so is $P\Delta P'$. □

Corollary 1.7.1. *Given a signed measure ν on $(X, \mathcal{M})$, consider the Hahn decomposition P, N of X and define the measures ν^+, ν^- as*

$$\nu^+(E) = \nu(E \cap P) \geq 0, \qquad \nu^-(E) = -\nu(E \cap N) \geq 0.$$

Then, we have the decomposition of ν as the difference of the non-negative measures ν^+, ν^- as

$$\nu = \nu^+ - \nu^-.$$

Definition 1.5. Let μ, ν be two measures on $(X, \mathcal{M})$. We say that μ and ν are mutually singular, denoted $\mu \perp \nu$, if there exists $E, F \in \mathcal{M}$ such that $E \cup F = X$, $E \cap F = \emptyset$, E is a null set for μ, and F is a null set for ν.

Example. For any signed measure ν, the corresponding Hahn decomposition P, N of X also gives the decomposition $\nu = \nu^+ - \nu^-$ where ν^+, ν^- are positive measures. Then, $\nu^+ \perp \nu^-$, with N being a null set for ν^+, P for ν^-.

Theorem 1.8 (Jordan Decomposition)**.** *Let ν be a signed measure on $(X, \mathcal{M})$. Then, there exist unique positive measures ν^+, ν^- on $\mathcal{M}$ such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.*

Proof. We have already shown that every signed measure ν admits such ν^+, ν^- via the Hahn decomposition P, N of X. Thus, it is enough to show that if $\nu = \mu^+ - \mu^-$ for positive measures μ^+, μ^-, and $\mu^+ \perp \mu^-$ with E, F being null sets for μ^+, μ^-, then $\mu^+ = \nu^+$ and $\mu^- = \nu^-$.

Let $A \in \mathcal{M}$. Then,

$$0 \leq \mu^+(A) = \mu^+(A \cap F) + \mu^+(A \cap E) = \mu^+(A \cap F)$$

since $A \cap E \subseteq E$ which is a null set for μ^+. Additionally, $A \cap F \subseteq F$ which is a null set for μ^-, so

$$0 \leq \mu^+(A \cap F) = \mu^+(A \cap F) - \mu^+(A \cap F) = \nu(A \cap F).$$

This shows that every subset of F has positive ν-measure, i.e. F is a positive set for ν. Similarly,

$$0 \geq -\mu^-(A) = -\mu^-(A \cap E) = -\nu(A \cap E),$$

which shows that E is a negative set for ν. Thus, F, E is a Hahn decomposition of X. Theorem 1.7 immediately tells us that $P \Delta F = N \Delta E$ is a null set for ν. Now,

$$\mu^+(A) = \nu(A \cap F) = \nu(A \cap F \cap P) + \nu(A \cap F \cap N),$$

but $A \cap F \cap N \subseteq (P \cap E) \cup (F \cap N) = P \Delta F$ which is a null set for ν. Thus,

$$\mu^+(A) = \nu(A \cap F \cap P).$$

Using the same arguments,

$$\nu^+(A) = \nu(A \cap P) = \nu(A \cap P \cap F) + \nu(A \cap P \cap E),$$

but $A \cap P \cap E \subseteq (P \cap E) \cup (F \cap N) = P \Delta F$, hence

$$\nu^+(A) = \nu(A \cap F \cap P) = \mu^+(A).$$

Thus, $\nu^+ = \mu^+$. An analogous argument shows that $\nu^- = \mu^-$. □

Remark. The Hahn and Jordan decomposition theorems together give the existence of the decomposition of any signed measure $\nu = \nu^+ - \nu^-$ along with its uniqueness in a certain sense.

1.3 The total variation measure

Definition 1.6. Let ν be a signed measure on $(X, \mathcal{M})$, and let $\nu = \nu^+ - \nu^-$ for positive measures ν^+, ν^- with $\nu^+ \perp \nu^-$. Then, the total variation measure $|\nu|$ of ν is defined as

$$|\nu| = \nu^+ + \nu^- \geq 0.$$

Lemma 1.9. *Let $E \in \mathcal{M}$. The following are equivalent.*

1. *E is a null set for ν.*
2. *$\nu^+(E) = \nu^-(E) = 0$.*
3. *$|\nu|(E) = 0$.*

Proof. Let P, N be the Hahn decomposition of X associated with ν.

(*1⇒2*) Note that $E \cap N, E \cap P \subseteq E$ are null sets for ν. Thus,

$$\begin{aligned}\nu^+(E) &= \nu^+(E \cap P) + \nu^+(E \cap N) \\ &= \nu^+(E \cap P) \\ &= \nu^+(E \cap P) - \nu^-(E \cap P) \\ &= \nu(E \cap P) \\ &= 0.\end{aligned}$$

Similarly,

$$\begin{aligned}\nu^-(E) &= \nu^-(E \cap P) + \nu^-(E \cap N) \\ &= \nu^-(E \cap N) \\ &= -\nu^+(E \cap N) + \nu^-(E \cap N) \\ &= -\nu(E \cap N) \\ &= 0.\end{aligned}$$

(*2⇒3*) follows trivially.

(*3⇒1*) By the positivity of ν^+, ν^-, we immediately have $\nu^+(E) = \nu^-(E) = 0$. Thus, for any measurable $F \subseteq E$, we have $\nu^+(F) = \nu^-(F) = 0$, hence $\nu(F) = 0$. Therefore, E is a null set for ν. □

Lemma 1.10. *Let ν, μ be signed measures. The following are equivalent.*

1. $\nu \perp \mu$.
2. $\nu^+ \perp \mu$ *and* $\nu^- \perp \mu$
3. $|\nu| \perp \mu$.

Proof. Let P, N be the Hahn decomposition associated with ν.

(*1⇒2*) Let $\nu \perp \mu$ via the decomposition E, F of X. Then F is a null set for ν, E for μ. We claim that $\nu^+ \perp \mu$ via the decomposition $N \cup F, P \cap E$. It is clear that $P \cap E \subseteq E$ is a null set for μ; we now show that $N \cup F$ is a null set for ν^+, i.e. $\nu^+(N \cup F) = 0$. Indeed,

$$0 \le \nu^+(N \cup F) \le \nu^+(N) + \nu^+(F) = \nu^+(F) = 0.$$

We have used the fact that N is a null set for ν^+, and F is a null set for ν together with the previous lemma.

The proof that $\nu^- \perp \mu$ via the decomposition $N \cup F, P \cap E$ is analogous.

(*2⇒3*) Let $\nu^+ \perp \mu$ via E_1, F_1, and $\nu^- \perp \mu$ via E_2, F_2. We claim that $|\nu| \perp \mu$ via the decomposition $E_1 \cup E_2, F_1 \cap F_2$. Is is clear that $E_1 \cup E_2$ is a null set for μ since it is the union of the null sets E_1, E_2 for μ. We must show that $F_1 \cap F_2$ is a null set for $|\nu|$, i.e. $|\nu|(F_1 \cap F_2) = 0$; but this is clear from the fact that F_1, F_2 are null sets for ν^+, ν^-, hence

$$0 \le |\nu|(F_1 \cap F_2) = \nu^+(F_1 \cap F_2) + \nu^-(F_1 \cap F_2) \le \nu^+(F_1) + \nu^-(F_2) = 0.$$

(*3⇒1*) Let $|\nu| \perp \mu$ via E, F. We claim that $\nu \perp \mu$ via the same decomposition E, F; indeed $|\nu|(F) = 0$ immediately shows that F is a null set for ν by the previous lemma. □

Lemma 1.11. *Let ν be a signed measure.*

1. *ν is a finite measure if and only if $|\nu|$ is a finite measure.*
2. *ν is a σ-finite measure if and only if $|\nu|$ is a σ-finite measure.*

Proof. The finiteness of ν implies that ν^+ and ν^- are finite hence their sum $|\nu|$ is finite, and vice versa. Next, if X is a countable union of $\{X_i\}$ each of which has finite ν-measure, each $\nu^+(X_i) < \infty$, $\nu^-(X_i) < \infty$ i.e. each X_i has finite $|\nu|$-measure, and vice versa. □

Definition 1.7. The space of functions $L^1(\nu)$, where ν is a signed measure, is defined by

$$L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-).$$

Lemma 1.12. *The spaces $L^1(\nu) = L^1(|\nu|)$.*

Proof. Note that

$$\int_P |f|\, d\nu = \int_X |f|\, d\nu^+ = \int_P |f|\, d|\nu|, \qquad \int_N |f|\, d\nu = -\int_X |f|\, d\nu^- = -\int_N |f|\, d|\nu|.$$ □

Definition 1.8. Let ν be a signed measure and μ be a positive measure. We say that ν is absolutely continuous with respect to μ, denoted $\nu \ll \mu$, if $\mu(E) = 0 \implies \nu(E) = 0$ where $E \in \mathcal{M}$.

Remark. If $\nu \ll \mu$, then all null sets for μ are null sets for ν.

Remark. If $\nu \ll \mu$ and $\nu \perp \mu$, then $\nu = 0$. Indeed, if $X = E \cup F$, $E \cap F = 0$ with F null for ν, E for μ, then for any $A \in \mathcal{M}$, we have $\nu(A) = \nu(A \cap E)$. But $A \cap E \subseteq E$ is a null set for μ, hence $\mu(A \cap E) = 0 \implies \nu(A) = \nu(A \cap E) = 0$.

Lemma 1.13. *Let ν be a signed measure and let μ be a positive measure. The following are equivalent.*

1. *$\nu \ll \mu$.*
2. *$\nu^+ \ll \mu$ and $\nu^- \ll \mu$.*
3. *$|\nu| \ll \mu$.*

Proof. *(1⇒2)* If $\mu(E) = 0$, then $\mu(F) = 0$ for every measurable $F \subseteq E$, hence $\nu(F) = 0$. In other words, E is a null set for ν, hence $\nu^+(E) = \nu^-(E) = 0$ by a previous lemma.

(2⇒3) If $\mu(E) = 0$, then $\nu^+(E) = \nu^-(E) = 0$, hence $|\nu|(E) = \nu^+(E) - \nu^-(E) = 0$.

(3⇒1) If $\mu(E) = 0$, then $|\nu|(E) = 0$ hence E is a null set for ν by a previous lemma, giving $\nu(E) = 0$. □

Lemma 1.14. *Let ν, μ be finite measures. Then, $\nu \ll \mu$ if and only if the following holds: for every $\epsilon > 0$, there exists $\delta > 0$ such that whenever $\mu(E) < \delta$, we have $|\nu(E)| < \epsilon$.*

Proof. ($\Leftarrow$) If $\mu(E) = 0$, then for any $\epsilon > 0$, we must have $\delta > 0$ such that $\mu(E) < \delta \implies |\nu(E)| < \epsilon$. This forces $\nu(E) = 0$, hence $\nu \ll \mu$.

($\Rightarrow$) Let $\nu \ll \mu$. Suppose that there exists $\epsilon > 0$ such that for all $\delta > 0$, we have sets E_δ where $\mu(E_\delta) < \delta$ but $\nu(E_\delta) \geq \epsilon$. Set $\delta = 2^{-n}$, and obtain corresponding E_n where $\mu(E_n) < 2^{-n}$, $\nu(E_n) \geq \epsilon$. Now, set

$$F_n = \bigcup_{i=n}^{\infty} E_n, \qquad F = \bigcap_{i=1}^{\infty} F_i.$$

Then, each $F_n \supseteq F_{n+1}$, and

$$\mu(F_n) \leq \sum_{i=n}^{\infty} \frac{1}{2^i} = \frac{2}{2^n} \to 0.$$

Thus, continuity from above with the finiteness of μ gives

$$\mu(F) = \lim_{n\to\infty} \mu(F_n) = 0.$$

But each $\nu(F_n) \geq \nu(E_n) \geq \epsilon$, hence $\nu(F) \geq \epsilon$ by the finiteness of ν, a contradiction. □

Corollary 1.14.1. *Let $f \in L^1(\mu)$. Then for all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $\mu(E) < \delta$, we have*

$$\left| \int_E f \, d\mu \right| < \epsilon.$$

1.4 The Radon-Nikodym theorem

Theorem 1.15 (Radon-Nikodym)**.** *Let ν be a σ-finite measure, and let μ be a σ-finite positive measure. Then, we have the following.*

1. *There exists a unique decomposition $\nu = \nu_1 + \nu_2$ such that $\nu_1 \ll \mu$ and $\nu_2 \perp \mu$. Both ν_1 and ν_2 are σ-finite.*
2. *There exists a measurable, μ-integrable (in the extended sense) function f such that*

$$\nu_1(E) = \int_E f \, d\mu.$$

Furthermore, if two such functions satisfy the above, they must be equal μ-almost everywhere.

Example. Let m be the Lebesgue measure and let c be the counting measure on $\mathbb{R}$. Clearly, $m \ll c$. If there existed a function f such that

$$m(E) = \int_E f \, dc,$$

then we would have

$$0 = m(\{x\}) = \int_{\{x\}} f \, dc = f(x)$$

for each $x \in \mathbb{R}$, forcing $f = 0$ hence $m = 0$ a contradiction.

Definition 1.9. Let ν be a σ-finite measure, and let μ be a positive σ-finite measure, such that $\nu \ll \mu$. Using the Radon-Nikodym theorem, we can pick a measurable function f such that

$$\nu(E) = \int_E f \, d\mu$$

for all $E \in \mathcal{M}$. Then, the function

$$f \equiv \frac{d\nu}{d\mu}$$

is called the Radon-Nikodym derivative of ν with respect to μ.

Remark. If $d\nu = f \, d\mu$, then $d|\nu| = |f| \, d\mu$.

Lemma 1.16. *Let $\nu \ll \mu$ and $\mu \ll \lambda$. Then,*

$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$$

λ-almost everywhere.

2 Complex measures

2.1 Basic definitions

Definition 2.1. Let $\mathcal{M}$ be a σ-algebra over X. A function $\nu\colon \mathcal{M} \to \mathbb{C}$ is called a complex measure if

1. $\nu(\emptyset) = 0$.
2. For all countable collections of disjoint measurable sets $\{E_i\}$, we have

$$\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \nu(E_i).$$

Remark. A complex measure only assumes finite values.

Remark. A complex measure ν can be decomposed as

$$\nu = \nu_r + i\nu_i$$

where ν_r, ν_i are signed measures.

Definition 2.2. A function $f \in L^1(\nu)$ if and only if $f \in L^1(\nu_r) \cap L^1(\nu_i)$.

Lemma 2.1. *Let ν, μ be complex measures. The following are equivalent.*

1. $\nu \perp \mu$.
2. $\nu_r \perp \mu_r$ *and* $\nu_i \perp \mu_i$.

Lemma 2.2. *Let ν be a complex measure, and let μ be a positive measure. The following are equivalent.*

1. $\nu \ll \mu$.
2. $\nu_r \ll \mu_r$ *and* $\nu_i \ll \mu_i$.

2.2 The Radon-Nikodym theorem and the total variation measure

Theorem 2.3 (Radon-Nikodym)**.** *Let ν be a complex measure, and let μ be a σ-finite positive measure. Then, there exists a unique decomposition*

$$\nu = \nu_1 + \nu_2$$

such that $\nu_1 \ll \mu$, $\nu_2 \perp \mu$. There also exists a function $f \in L^1(\mu)$ such that

$$\nu_1(E) = \int_E f \, d\mu.$$

Furthermore, f us unique μ-almost everywhere.

Lemma 2.4. *Let ν be a complex measure and let μ_1, μ_2 be σ-finite positive measures, such that for $f_1 \in L^1(\mu_1)$, $f_2 \in L^1(\mu_2)$,*

$$d\nu = f_1\, d\mu_1 = f_2\, d\mu_2.$$

Then,

$$|f_1|\, d\mu_1 = |f_2|\, d\mu_2.$$

Definition 2.3. Let ν be a complex measure, and let μ be a positive measure such that $d\nu = f\, d\mu$. Then, $d|\nu| = |f|\, d\mu$ defines a measure $|\nu|$ called the total variation measure.

Remark. Such a measure exists since $\nu \ll \nu_r^+ + \nu_r^- + \nu_i^+ + \nu_i^-$ supplies us with a Radon-Nikodym derivative f.

Remark. This definition is independent of our choice of μ, hence f by the previous lemma.

Lemma 2.5.

$$|\nu(E)| \leq |\nu|(E).$$

Lemma 2.6.

$$|\nu| = \inf\{\lambda : |\nu(E)| \leq \lambda(E) \text{ for all } E \in \mathcal{M}\}.$$

Lemma 2.7.

$$\nu \ll |\nu|, \qquad \left|\frac{d\nu}{d|\nu|}\right| = 1 \quad |\nu|\text{-almost everywhere.}$$

Lemma 2.8.

$$L^1(\nu) = L^1(|\nu|).$$

Lemma 2.9.

$$|\int f\, d\nu| \leq \int |f|\, d|\nu|.$$

Lemma 2.10.

$$|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|.$$

Theorem 2.11. *Let $\mathscr{M}(X)$ be the collection of all complex measures on $(X, \mathcal{M})$. Then, given $\nu_1, \nu_2 \in \mathscr{M}(X)$, we have $\nu_1 + \nu_2 \in \mathscr{M}(X)$ and $\alpha\nu_1 \in \mathscr{M}(X)$ for all $\alpha \in \mathbb{C}$. Thus, $\mathscr{X}(X)$ is a vector space. Furthermore,*

$$\|\nu\| = |\nu|(X)$$

defines a norm on $\mathscr{M}(X)$. With this, $\mathscr{M}(X)$ is a Banach space.

3 Differentiation on Euclidean Spaces

3.1 Locally integrable functions

Definition 3.1. Let $f \colon \mathbb{R}^n \to \mathbb{C}$ be measurable. We say that f is locally integrable if for every bounded measurable $E \subseteq \mathbb{R}^n$,

$$\int_E |f| \, dm$$

is finite. The class of all such functions is denoted $L^1_{\text{loc}}(\mathbb{R}^n)$.

Definition 3.2. For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, define the average value of f on $B_r(x)$ as

$$A_r(f)(x) = \frac{1}{m(B_r(x))} \int_{B_r(x)} f \, dm.$$

Lemma 3.1. *$A_r(f)(x)$ is jointly continuous in r and x.*

Definition 3.3. For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, define its Hardy-Littlewood maximal function as

$$Hf(x) = \sup_{r>0} A_r(|f|)(x) = \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f| \, dm.$$

3.2 The Maximal Function Theorem

Lemma 3.2. *Let $\mathscr{C}$ be a collection of open balls in $\mathbb{R}^n$, and let*

$$U = \bigcup_{B \in \mathscr{C}} B.$$

If $c < m(U)$, then there exist disjoint $B_1, \dots, B_k \in \mathscr{C}$ such that

$$c < 3^n \sum_{i=1}^{k} m(B_i).$$

Theorem 3.3 (Maximal Function)**.** *There exists a constant $C > 0$ such that for all $f \in L^1$ and all $\alpha > 0$,*

$$m(\{x : Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \int |f| \, dm.$$

Theorem 3.4. *If $f \in L^1_{loc}(\mathbb{R}^n)$, then*

$$\lim_{r \to 0} A_r(f)(x) = f(x).$$

almost everywhere on $\mathbb{R}^n$.

3.3 The Lebesgue Differentiation Theorem

Definition 3.4. For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, define the Lebesgue set L_f of f as the collection of all $x \in \mathbb{R}^n$ such that

$$\lim_{r \to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(x) - f(x')| \, dx' = 0.$$

Lemma 3.5. *For $f \in L^1_{loc}(\mathbb{R}^n)$, we have $m(L_f^c) = 0$.*

Definition 3.5. We say that the family of measurable sets $\{E_r\}$ shrink nicely to $x \in \mathbb{R}^n$ if each $E_r \subseteq B_r(x)$, and there exists $\alpha > 0$ such that each $m(E_r) > \alpha m(B_r(x))$.

Remark. It is possible that none of the sets E_r contains x!

Theorem 3.6 (Lebesgue Differentiation)**.** *Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then for all $x \in L_f$, we have*

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} |f(x) - f(x')| \, dx' = 0, \qquad \lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f \, dm = f(x)$$

for all families $\{E_r\}$ which shrink nicely to x.

Definition 3.6. A positive Borel measure ν on $\mathbb{R}^n$ is called regular if the following hold.

1. $\nu(K)$ is finite for every compact $K \subseteq \mathbb{R}^n$.
2. $\nu(E) = \inf\{\nu(U) : E \subseteq U, U \subseteq \mathbb{R}^n \text{ is open}\}$ for every Borel measurable E.

Remark. The second condition follows from the first.

Theorem 3.7. *Let ν be a regular signed or complex Borel measure on $\mathbb{R}^n$, whose Radon-Nikodym representation is given by*

$$d\nu = f\, dm + d\lambda.$$

Then, for every family $\{E_r\}$ that shrinks nicely to $x \in \mathbb{R}^n$, we have

$$\lim_{r \to 0} \frac{\nu(E_r)}{m(E_r)} - f(x)$$

almost everywhere with respect to m.

4 Total and Bounded Variation

4.1 Total variation

Theorem 4.1. *Let $F \colon \mathbb{R} \to \mathbb{R}$ be increasing, and let*

$$G(x) = F(x^+) = \lim_{t \to x^+} F(x).$$

Then, the following hold.

1. *G is increasing and right continuous.*
2. *The set of discontinuities of F is countable.*
3. *Both F, G are differentiable almost everywhere, with $F' = G'$ almost everywhere.*

Definition 4.1. Let $F \colon \mathbb{R} \to \mathbb{R}$. The total variation function T_F of F is defined as

$$T_F(x) = \sup \left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : -\infty < x_0 < \cdots < x_n = x \right\}.$$

Lemma 4.2.

$$T_F(b) - T_F(a) = \sup \left\{ \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| : a = x_0 < \cdots < x_n = b \right\}.$$

Lemma 4.3. *The total variation function T_F of F is increasing, taking values in $[0, \infty]$.*

4.2 Bounded variation

Definition 4.2. We say that F is of bounded variation if

$$T(\infty) = \lim_{x\to\infty} T(x)$$

is finite. The class of such functions is denoted $BV(\mathbb{R})$.

Furthermore, the class of functions F such that $T_F(b) - T_F(a)$ is finite is denoted $BV[a,b]$.

Example. If F is bounded and increasing, then $F \in BV(\mathbb{R})$, with

$$T_F(x) = F(x) - F(-\infty).$$

Example. If $F, G \in BV(\mathbb{R})$, then $\alpha F + \beta G \in BV(\mathbb{R})$ for all $\alpha, \beta \in \mathbb{C}$.

Example. If F is differentiable and F' is bounded, then $F \in BV[a,b]$.

Lemma 4.4. *If $F \in BV(\mathbb{R})$, then $T_F \pm F$ are bounded and increasing functions.*

Theorem 4.5. *The following results hold.*

1. *$F \in BV(\mathbb{C})$ if and only if $\Re(F), \Im(F) \in BV(\mathbb{R})$.*
2. *$F \in BV(\mathbb{R})$ if and only if F is the difference of two bounded and increasing functions. Note that*
$$F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F).$$
3. *If $F \in BV(\mathbb{R})$, all of the following limits exist.*
$$F(x^+) = \lim_{t\to x^+} F(x), \qquad F(x^-) = \lim_{t\to x^-} F(x), \qquad F(\pm\infty) = \lim_{x\to\pm\infty} F(x).$$
4. *If $F \in BV(\mathbb{R})$, then the set of points on which F is discontinuous is countable.*
5. *If $F \in BV(\mathbb{R})$, and $G(x) = F(x^+)$, then $F' = G'$ almost everywhere.*

Definition 4.3. For $F \in BV(\mathbb{R})$, the following is called the Jordan representation of F.

$$F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F).$$

The two terms are called the positive and negative variations of F respectively.

Lemma 4.6. *Denote the positive and negative parts of x as $x^{\pm} = \frac{1}{2}(|x| \pm x)$. Then,*

$$\frac{1}{2}(T_F \pm F)(x) = \sup\left\{\sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]^{\pm} : -\infty < x_0 < \cdots < x_n = x\right\}.$$

4.3 Normalized Bounded Variation

Definition 4.4. Denote

$$T_F(-\infty) = \lim_{x \to -\infty} T_F(x).$$

We say that $F \in BV(\mathbb{R})$ is of normalized bounded variation if F is right continuous and $T_F(-\infty)$ is finite. The class of such functions is denoted $NBV(\mathbb{R})$.

Remark. If $F \in BV(\mathbb{R})$, then $G(x) = F(x^+) - F(-\infty)$ is in $NBV(\mathbb{R})$.

Lemma 4.7. *If $F \in NBV(\mathbb{C})$, then $T_F(-\infty) = 0$. Furthermore, if F is right continuous, so is T_F.*

Theorem 4.8. *Let μ be a complex Borel measure on $\mathbb{R}^n$. Let*

$$F(x) = \mu(-\infty, x].$$

Then, $F \in NBV(\mathbb{C})$.

Conversely, if $F \in NBV(\mathbb{C})$, there exists a complex Borel measure μ_F such that

$$\mu_F(-\infty, x] = F(x).$$

Furthermore,

$$|\mu_F| = \mu_{T_F}.$$

Corollary 4.8.1. *If $F \in NBV$, then $T_F \in NBV$.*

4.4 Fundamental Theorem of Calculus

Definition 4.5. We say that $F \colon \mathbb{R} \to \mathbb{C}$ is absolutely continuous if for every $\epsilon > 0$, there exists $\delta > 0$ such that for any finite disjoint intervals $\{(a_i, b_i)\}_{i=1}^n$ where $\sum_{i=1}^n (b_i - a_i) < \delta$, we have

$$\sum_{i=1}^{n} |F(b_i) - F(a_i)| < \epsilon.$$

Remark. All absolutely continuous functions are uniformly continuous.

Lemma 4.9. *Let $F \in NBV$. Then, F is absolutely continuous if and only if $\mu_F \ll m$.*

Corollary 4.9.1. *Let $f \in L^1(m)$. The function F defined by*

$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$

is in NBV, is absolutely continuous, and $f = F'$ almost everywhere.

Conversely, if $F \in NBV$, then $F' \in L^1(m)$, and

$$F(x) = \int_{-\infty}^{x} F'(t)\, dt.$$

Theorem 4.10 (Fundamental Theorem of Calculus)**.** *Let $-\infty < a < b < \infty$, and let $F\colon [a,b] \to \mathbb{C}$. The following statements are equivalent.*

1. *F is absolutely continuous on $[a,b]$.*
2. *There exists $f \in L^1([a,b], m)$ such that*

$$F(x) - F(a) = \int_{a}^{x} f(t)\, dt.$$

3. *F is differentiable almost everywhere, with $F' \in L^1([a,b], m)$, and*

$$F(x) - F(a) = \int_{a}^{x} F'(t)\, dt.$$

Lemma 4.11. *If F is absolutely continuous on $[a,b]$, then $F \in BV[a,b]$.*

Remark. The converse is not true: consider the Cantor function.

4.5 Integration by parts

Theorem 4.12. *Let $F, G \in NBV$, with at least one of them continuous. Then, for $-\infty < a < b < \infty$, we have*

$$\int_{(a,b]} F\, dG + \int_{(a,b]} G\, dF = F(b)G(b) - F(a)G(a).$$

Remark. Here, we denote $dF = d\mu_F = F'\, dm$.

5 Radon measures

Definition 5.1. A Radon measure μ on X is a Borel measure such that the following hold.

1. $\mu(K)$ is finite for all compact sets $K \subseteq X$.
2. μ is outer regular on Borel sets.
3. μ is inner regular on open sets.

5.1 Locally compact Hausdorff spaces

Lemma 5.1. *Let X be a locally compact Hausdorff space, let $U \subseteq X$ be open, and let $x \in U$. Then, there is a compact neighborhood of x contained in U.*

Lemma 5.2. *Let X be a locally compact Hausdorff space, and let U be open, K be compact, such that $K \subseteq U$. Then, there exists an open set V such that $\overline{V} is compact$, and $K \subseteq V \subseteq \overline{V} \subseteq U$.*

Lemma 5.3 (Urysohn)**.** *Let X be a locally compact Hausdorff space, let U be open, K be compact such that $K \subseteq U$. Then, there exists a continuous map $f\colon X \to [0,1]$, such that $f = 1$ on K, and $f = 0$ outside a compact subset of U.*

Lemma 5.4. *Let X be a locally compact Hausdorff space. The space $C_c(X)$, of compactly supported continuous functions on X, is dense in $C_0(X)$, the space of functions that vanish at infinity.*

Definition 5.2. Let X be a topological space, and let $E \subseteq X$. A partition of unity on E is a collection $\{h_\alpha\}_{\alpha \in I}$ of functions in $C(X, [0,1])$ such that the following hold.

1. Each $x \in X$ has a neighborhood on which only finitely many h_α's are non-zero.
2. For each $x \in E$, $\sum_{\alpha \in I} h_\alpha(x) = 1$.

Furthermore, we say that a partition of unity $\{h_\alpha\}_{\alpha \in I}$ is subordinate to an open cover $\mathscr{U}$ of E if for each $\alpha \in I$, there exists $U \in \mathscr{U}$ such that $\operatorname{supp}(h_\alpha) \subset U$.

Lemma 5.5. *Let X be a locally compact Hausdorff space, let $K \subseteq X$ be compact, and let $\{U_i\}_{i=1}^n$ be an open cover of K. Then, there exists a partition of unity on K subordinate to $\{U_i\}_{i=1}^n$, consisting of compactly supported functions.*

5.2 Positive linear functionals

Definition 5.3. A linear functional I on $C_c(X)$ is called positive if $I(f) \geq 0$ whenever $f \geq 0$.

Lemma 5.6. *Let I be a positive linear functional on $C_c(X)$. For any compact set $K \subseteq X$, there exists a constant $C_K > 0$ such that whenever $f \in C_c(X)$ with $\operatorname{supp}(f) \subseteq K$, we have*

$$|I(f)| \leq C_K \|f\|_\infty.$$

Definition 5.4. Let $U \subseteq X$ be open, and let $f \in C_c(X)$. We write $f \prec U$ when $0 \leq f \leq 1$, and $\operatorname{supp}(f) \subset U$.

Theorem 5.7 (Riesz Representation Theorem for $C_c(X)$)**.** *Let I be a positive linear functional on $C_c(X)$. Then, there exists a unique Radon measure μ on X such that*

$$I(f) = \int f \, d\mu,$$

and the following regularity conditions are satisfied.

1. *For all open $U \subseteq X$,*

$$\mu(U) = \sup\{I(f) : f \in C_c(X), f \prec U\}.$$

2. *For all compact $K \subseteq X$,*

$$\mu(K) = \inf\{I(f) : f \in C_c(X), f \geq \chi_K\}.$$

5.3 Regularity and approximation

Lemma 5.8. *Every Radon measure is inner regular on σ-finite sets.*

Corollary 5.8.1. *Any σ-finite Radon measure is regular.*

Corollary 5.8.2. *If X is σ-compact, then every Radon measure on X is regular.*

Lemma 5.9. *Let μ be a σ-finite Radon measure on X, and let $E \subseteq X$ be a Borel set.*

1. *For every $\epsilon > 0$, there exists open U and closed F, such that $F \subseteq E \subseteq U$ and $\mu(U \setminus F) < \epsilon$.*
2. *There exists an F_σ set A and a G_δ set B such that $A \subseteq E \subseteq B$ and $\mu(B \setminus A) = 0$.*

Theorem 5.10. *Let X be a locally compact Hausdorff space, where every open set is σ-compact. Then, every Borel measure on X that is finite on compact sets is regular, hence Radon.*

Lemma 5.11. *Let μ be a complex Borel measure. Then, μ is Radon if and only if $|\mu|$ is Radon.*

Additionally, if μ is a finite Borel measure, then μ us Radon if and only if the following property holds: for every Borel set E, given $\epsilon > 0$, there exists some compact set K and open set U such that $K \subseteq E \subseteq U$ and $\mu(U \setminus K) < \epsilon$.

Lemma 5.12. *Let μ be a Radon measure. Then, $C_c(X)$ is dense in $L^p(mu)$, where $1 \leq p < \infty$.*

Theorem 5.13 (Lusin)**.** *Let μ be a Radon measure, and let $f\colon X \to \mathbb{C}$ be a measurable function that vanishes outside a set of finite measure. Then, for any $\epsilon > 0$, there exists $\varphi \in C_c(X)$ such that $\varphi = f$ except on a set of measure less than ϵ. Furthermore, if f is bounded, then φ can be chosen such that $\|\varphi\|_\infty \leq \|f\|_\infty$.*

5.4 The dual of $C_0(X)$

Lemma 5.14. *Let $I \in C_0(X, \mathbb{R})^*$. Then, there exist positive linear functionals I^+, I^-, such that $I = I^+ - I^-$.*

Definition 5.5. We denote $\mathscr{M}(X)$ to be the vector space of all complex Radon measures on X. This is equipped with the norm

$$\|\mu\| = |\mu|(X).$$

Theorem 5.15 (Riesz Representation Theorem). *Let X be a locally compact Hausdorff space, let $\mu \in \mathcal{M}(X)$, and let $f \in C_0(X)$. Define*

$$I_\mu(f) = \int f \, d\mu.$$

Then, the map $\mu \to I_\mu$ is an isometric isomorphism from $\mathcal{M}(X)$ to $C_0(X)^*$.