MA4102

Functional Analysis

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Contents

1 Normed linear spaces

1.1 Basic definitions

Definition 1.1. Let X be a vector space over the field F (typically $\mathbb{R}$ or $\mathbb{C}$). A subset $S \subseteq X$ is called linearly independent if for every finitely many vectors $x_1, \dots, x_n \in S$, we have

$$\sum_{i=1}^{n} c_i x_i = 0 \implies c_i = 0 \text{ for each } 1 \leq i \leq n.$$

Definition 1.2. A subset $B \subseteq X$ is called a Hamel basis of X if it is linearly independent, and every element of X can be written as a finite linear combination of elements from B.

Example. The standard basis $\{\boldsymbol{e}_i\}$ of $\mathbb{R}^n$ is a Hamel basis.

Example. The polynomials $\{1, x, x^2, \dots\}$ is a Hamel basis of the space $\mathscr{P}(\mathbb{R})$ of all polynomials.

Definition 1.3. A norm on X is a map $\|\cdot\|: X \to [0, \infty)$ satisfying the following properties.

1. $\|x\| = 0$ if and only if $x = 0$.
2. $\|kx\| = |k|\|x\|$ for all $x \in X$, $k \in F$.
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

The space $(X, \|\cdot\|)$ is called a normed linear space.

Example. The vector space $\mathbb{R}^n$ equipped with the metric

$$\|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

is a normed linear space.

Example. The vector space of continuous functions $\mathcal{C}[0, 1]$ equipped with the supremum norm

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$$

is a normed linear space.

Example. The vector space of continually differentiable functions $\mathcal{C}^1[0, 1]$ equipped with the norm

$$\|f_1\| = \sup_{x \in [0,1]} |f(x)| + \sup_{x \in [0,1]} |f'(x)|$$

is a normed linear space.

Example. The function spaces $L^p(\mu)$ for $1 \leq p < \infty$, equipped with the metrics

$$\|f\|_p = \left(\int |f|^p \, d\mu\right)^{1/p}$$

are normed linear spaces.

Definition 1.4. Every normed linear space $(X, \|\cdot\|)$ can be equipped with the normed topology τ_d, induced by the metric

$$d(x, y) = \|x - y\|.$$

Lemma 1.1. *Let $x_n \to x$, $y_n \to y$ in X with the normed topology, and let $\alpha_n \to \alpha$ in X. Then,*

1. *$x_n + y_n \to x + y$ in X.*
2. *$\alpha_n x_n \to \alpha x$ in X.*

1.2 Banach spaces

Definition 1.5. A normed linear space X is called a Banach space if (X, d) is a complete metric space where

$$d(x, y) = \|x - y\|.$$

Example. The spaces $\mathbb{R}^n$ with the metrics $\|\cdot\|_p$ for $1 \leq p \leq \infty$ are Banach spaces.

Example. The sequence spaces ℓ^p for $1 \leq p \leq \infty$ are Banach spaces.

Definition 1.6. Let X be a normed linear space. A countable collection $\{x_i\} \subseteq X$ is called a Schauder basis of X if each $\|x_i\| = 1$, and every vector $x \in X$ can be uniquely written as

$$x = \sum_{i=1}^{\infty} c_i x_i$$

for $c_i \in F$.

Remark. This infinite sum represents a convergent limit of partial sums in X.

Remark. A Schauder basis is linearly independent, from the uniqueness of the expansion $0 = \sum_{i=1}^{\infty} 0 x_i$.

Lemma 1.2. *The space of continuous functions $\mathcal{C}[0, 1]$ equipped with the supremum norm*

$$\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$$

is a Banach space.

Proof. Let $\{f_n\}$ be a Cauchy sequence of functions in $\mathcal{C}[0, 1]$. We claim that this sequence converges to some f in $\mathcal{C}[0, 1]$, i.e. there exists $f \in \mathcal{C}[0, 1]$ such that $\|f_n - f\| \to 0$.

Using the fact that $\{f_n\}$ is Cauchy, we have the following: for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have

$$\|f_n - f_m\| < \epsilon.$$

In particular, for each $x \in X$,

$$|f_n(x) - f_m(x)| \leq \sup_{x \in [0,1]} |f_n(x) - f_m(x)| = \|f_n - f_m\| < \epsilon.$$

In other words, each of the real sequences $\{f_n(x)\}$ is Cauchy. From the completeness of $\mathbb{R}$, all such sequences converge, hence the pointwise limit

$$f : [0, 1] \to \mathbb{R}, \qquad x \mapsto \lim_{n \to \infty} f_n(x)$$

exists. Furthermore, each

$$\lim_{m\to\infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)| \le \epsilon,$$

hence

$$\|f_n - f\| = \sup_{x\in[0,1]} |f_n(x) - f(x)| \le \epsilon.$$

Thus, $\|f_n - f\| \to 0$, i.e. $f_n \to f$ in X.

Finally, we must show that $f \in \mathcal{C}[0,1]$, i.e. that f is continuous. Fix $x_0 \in X$, and let $\epsilon > 0$. Since $f_n \to f$ in X, pick $N \in \mathbb{N}$ such that for all $n \ge N$,

$$|f_n(x) - f(x)| \le \sup_{x\in[0,1]} |f_n(x) - f(x)| = \|f_N - f\| < \frac{\epsilon}{3}.$$

From the continuity of f_N, there exists $\delta > 0$ such that whenever $|x - x_0| < \delta$, we have

$$|f_N(x) - f_N(x_0)| < \frac{\epsilon}{3}.$$

Thus, whenever $|x - x_0| < \delta$, we have

$$|f(x) - f(x_0)| \le |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This shows that f is continuous at each $x_0 \in X$, as desired. □

Exercise 1.1. Is the space of continuous functions $\mathcal{C}[0,2]$ equipped with the norm

$$\|f\|_1 = \int_0^2 |f(x)|\,dx$$

a Banach space?

Solution. Consider the functions

$$f_n\colon [0,2] \to \mathbb{R}, \qquad x \mapsto \frac{x^n}{1+x^n}.$$

Note that this sequence has a pointwise limit $f_n \to f$, where

$$f\colon [0,2] \to \mathbb{R}, \qquad x \mapsto \begin{cases} 0, & \text{if } 0 \le x < 1, \\ 1/2, & \text{if } x = 1, \\ 1, & \text{if } 1 < x \le 2. \end{cases}$$

Now,

$$\begin{aligned} \int_0^2 |f_n(x) - f(x)|\,dx &= \int_0^1 \left|\frac{x^n}{1+x^n}\right| dx + \int_1^2 \left|\frac{x^n}{1+x^n} - 1\right| dx \\ &\le \int_0^1 |x^n|\,dx + \int_1^2 |x^{-n}|\,dx \\ &= \frac{1}{n+1} - \frac{1}{n-1}(2^{-n+1} - 1) \to 0. \end{aligned}$$

Thus, given $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that for all $n \ge N$,

$$\int_0^2 |f_n(x) - f(x)|\,dx < \frac{\epsilon}{2}.$$

Then, for all $m, n \geq N$, we have

$$\begin{aligned}\|f_n - f_m\|_1 &= \int_0^2 |f_n(x) - f_m(x)|\,dx \\ &\leq \int_0^2 |f_n(x) - f(x)|\,dx + \int_0^2 |f_m(x) - f(x)|\,dx \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

This shows that $\{f_n\}$ is Cauchy in $\mathcal{C}[0, 2]$. However, if $\|f_n - g\|_1 \to 0$ for some continuous function $g \in \mathcal{C}[0, 2]$, then

$$0 \leq \int_0^2 |f(x) - g(x)|\,dx \leq \int_0^2 |f(x) - f_n(x)|\,dx + \int_0^2 |f_n(x) - g(x)|\,dx \to 0,$$

whence

$$\int_0^2 |f(x) - g(x)|\,dx = 0.$$

In particular,

$$\int_{[0,1)} |g(x)|\,dx = 0, \qquad \int_{(1,2]} |1 - g(x)|\,dx = 0.$$

The continuity of g forces $g(x) = 0$ on $[0, 1)$ and $g(x) = 1$ on $(1, 2]$. Again, the continuity of g guarantees $\delta > 0$ such that for all $|x - 1| < \delta$, we have $|g(x) - g(1)| < 1/4$. But

$$1 = |g(1 + \delta/2) - g(1 - \delta/2)| \leq |g(1 + \delta/2) - g(1)| + |g(1) - g(1 - \delta/2)| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

a contradiction.

Thus, the above Cauchy sequence $\{f_n\}$ does not converge in $\mathcal{C}[0, 1]$, hence this is not a Banach space.

Lemma 1.3 (Young). *Let $a, b \geq 0$, and $1 \leq p \leq \infty$. Then,*

$$a^{1/p} b^{1/q} \leq \frac{a}{p} + \frac{b}{q}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Lemma 1.4 (Hölder). *Let $x, y \in \ell^p$ for $1 \leq p \leq \infty$. Then,*

$$\|x \cdot y\|_1 \leq \|x\|_p \|y\|_q, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Lemma 1.5 (Minkowski). *Let $x, y \in \ell^p$ for $1 \leq p \leq \infty$. Then,*

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Lemma 1.6. *Cauchy sequences in a normed linear space are bounded.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in the normed linear space X. Then, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have

$$\|x_n - x_m\| \leq 1.$$

In particular, putting $m = N$, we have for all $n \geq N$,

$$\|x_n\| = \|x_n - x_N + x_N\| \leq \|x_n - x_N\| + \|x_N\| \leq 1 + \|x_N\|.$$

Thus, for all $n \in \mathbb{N}$,

$$\|x_n\| \leq \max\{\|x_1\|, \dots, \|x_{N-1}\|, 1 + \|x_N\|\}.$$ □

Lemma 1.7. *The spaces of sequences $\ell^p(\mathbb{R})$ for $1 \leq p < \infty$ are Banach spaces.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in ℓ^p. Note that each term is of the form

$$x_n = (x_n^1, x_n^2, \dots, x_n^k, \dots).$$

Given $\epsilon > 0$, we can pick $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$|x_n^k - x_m^k| \leq \left(\sum_{i=1}^{\infty} |x_n^i - x_m^i|^p\right)^{1/p} = \|x_n - x_m\|_p \leq \epsilon.$$

This shows that the sequences $\{x_n^k\}_{n \in \mathbb{N}}$ for each $k \in \mathbb{N}$ are Cauchy in $\mathbb{R}$. By the completeness of $\mathbb{R}$, they converge to some $x_n^k \to x^k$. Set

$$x = (x^1, x^2, \dots, x^k, \dots).$$

First, we show that $x \in \ell^p$. Recall that Cauchy sequences in a normed linear space are bounded, hence there exists $M > 0$ such that for each $n \in \mathbb{N}$,

$$\sum_{i=1}^{\infty} |x_n^i|^p = \|x_n\|_p^p < M.$$

Thus, for every $k \in \mathbb{N}$, the partial sum

$$\sum_{i=1}^{k} |x_n^i|^p < M.$$

Taking the limit $n \to \infty$,

$$\lim_{n\to\infty} \sum_{i=1}^{k} |x_n^i|^p = \sum_{i=1}^{k} \lim_{n\to\infty} |x_n^i|^p = \sum_{i=1}^{\infty} |x^i|^p \leq M.$$

In other words, each partial sum is bounded, hence taking the limit $k \to \infty$,

$$\lim_{k\to\infty} \sum_{i=1}^{k} |x^i|^p = \sum_{i=1}^{\infty} |x_i|^p = \|x\|_p^p \leq M.$$

Finally, we show that $x_n \to x$ in ℓ^p, i.e. that $\|x_n - x\|_p \to 0$. Note that given $\epsilon > 0$, we have for all $n, m \geq N$,

$$\sum_{i=1}^{\infty} |x_n^i - x_m^i|^p \leq \epsilon^p,$$

hence for all $k \in \mathbb{N}$, the partial sums

$$\sum_{i=1}^{k} |x_n^i - x_m^i|^p \leq \epsilon^p.$$

Thus,

$$\lim_{m\to\infty} \sum_{i=1}^{k} |x_n^i - x_m^i|^p = \sum_{i=1}^{k} \lim_{m\to\infty} |x_n^i - x_m^i|^p = \sum_{i=1}^{k} |x_n^i - x^i| \leq \epsilon^p.$$

Taking the limit of partial sums,

$$\lim_{k\to\infty} \sum_{i=1}^{k} |x_n^i - x^i|^p = \sum_{i=1}^{\infty} |x_n^i - x^i| = \|x_n - x\|_p^p \leq \epsilon^p.$$ □

Exercise 1.2. Show that the space of sequences $\ell^\infty(\mathbb{R})$ is a Banach space.

Solution. Let $\{x_n\}$ be a Cauchy sequence in ℓ^∞. Then for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$|x_n^k - x_m^k| \leq \sup_{k\in\mathbb{N}} |x_n^k - x_m^k| = \|x_n - x_m\|_\infty < \epsilon.$$

This shows that the sequences $\{x_n^k\}_{n\in\mathbb{N}}$ are Cauchy in $\mathbb{R}$. By the completeness of $\mathbb{R}$, they converge to some $x_n^k \to x^k$. Set

$$x = (x^1, x^2, \dots, x^k, \dots).$$

First, we show that $x \in \ell^\infty$. Since Cauchy sequences in a normed linear space are bounded, there exists $M > 0$ such that for each $n \in \mathbb{N}$,

$$|x_n^k| \leq \sup_{i\in\mathbb{N}} |x_n^i| = \|x_n\|_\infty < M.$$

Thus, taking the limit $n \to \infty$,

$$\lim_{n\to\infty} |x_n^k| = |x^k| \leq M.$$

This shows that

$$\sup_{k\in\mathbb{N}} |x^k| = \|x\|_\infty \leq M.$$

Finally, we show that $x_n \to x$ in ℓ^∞, i.e. $\|x_n - x\|_\infty \to 0$. Note that given $\epsilon > 0$, we have for all $m, n \geq N$,

$$|x_n^k - x_m^k| \leq \|x_n - x_m\|_\infty < \epsilon$$

for each $k \in \mathbb{N}$. Thus, taking the limit $m \to \infty$,

$$\lim_{m\to\infty} |x_n^k - x_m^k| = |x_n^k - x^k| \leq \epsilon.$$

This shows that

$$\sup_{k\in\mathbb{N}} |x_n^k - x^k| = \|x_n - x\|_\infty \leq \epsilon.$$

Lemma 1.8. *Let $x \in \mathbb{R}^n$. Then $\|x\|_p \to \|x\|_\infty$ as $p \to \infty$.*

Proof. Note that for each $1 \leq k \leq n$,

$$\|x\|_p^p = \sum_{i=1}^n |x^i|^p \geq |x^k|^p.$$

Thus,

$$\|x\|_p \geq \max_{1\leq k\leq n} |x^k| = \|x\|_\infty.$$

Next, note that

$$\|x\|_p^p = \sum_{i=1}^n |x^i|^p = \|x\|_\infty^p \left(\sum_{i=1}^n \frac{|x^i|^p}{\|x\|_\infty^p} \right) \leq \|x\|_\infty^p \cdot n,$$

hence

$$\|x\|_p \leq \|x\|_\infty \cdot n^{1/p}.$$

Taking the limit $p \to \infty$, we have

$$\|x\|_p \leq \|x\|_\infty.$$

□

Lemma 1.9. *Let $x \in \ell^q(\mathbb{R})$ for some $q \geq 1$. Then $\|x\|_p \to \|x\|_\infty$ as $p \to \infty$.*

Proof. Let $p \geq 1$. Note that for each $k \in \mathbb{N}$,

$$\|x\|_p^p = \sum_{i=1}^\infty |x^i|^p \geq \sum_{i=1}^k |x^i|^p \geq |x^k|^p.$$

Thus,

$$\|x\|_p \geq \sup_{k\in\mathbb{N}} |x^k| = \|x\|_\infty.$$

This immediately shows that if $\|x\|_\infty = \infty$, then $\|x\|_p = \infty$. Thus, we can assume that $\|x\|_\infty < \infty$.

Let $p \geq q \geq 1$. Then by Hölder's inequality,

$$\|x\|_p^p = \sum_{i=1}^\infty |x^i|^{p-q} \cdot |x^i|^q \leq \|x\|_\infty^{p-q} \cdot \sum_{i=1}^\infty |x^i|^q.$$

In other words,

$$\|x\|_p \leq \|x\|_\infty^{1-q/p} \cdot \|x\|_q^{q/p}.$$

Note that by assumption, $\|x\|_q < \infty$. Taking the limit as $p \to \infty$, we now have

$$\|x\| \leq \|x\|_\infty.$$

□

1.3 Linear maps

Definition 1.7. Let V, W be normed linear spaces over $\mathbb{R}$ or $\mathbb{C}$. Let $T\colon V \to W$ be a linear map. Then, T is called a bounded linear map if T is a continuous map between the normed topological spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$.