MA3205

Geometry of Curves and Surfaces

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Contents

1 Curves

1.1 Introduction

Definition 1.1. A curve is a continuous map $\gamma\colon \mathbb{R} \to \mathbb{R}^n$.

Definition 1.2. A smooth curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ is C^∞, i.e. differentiable arbitrarily times.

Definition 1.3. A closed curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ is periodic, i.e there exists some c such that $\gamma(t + c) = \gamma(t)$ for all $t \in \mathbb{R}$.

Example. Alternatively, a closed curve can be thought of as a continuous map $\gamma\colon S^1 \to \mathbb{R}^n$. For instance, given a closed curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ with period c, we can define the corresponding map

$$\tilde{\gamma}\colon S^1 \to \mathbb{R}^n, \qquad \tilde{\gamma}(e^{it}) = \gamma(ct/2\pi).$$

Definition 1.4. A simple curve $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ is injective on its period.

Theorem 1.1 (Four Vertex Theorem)**.** *The curvature of a simple, closed, smooth plane curve has at least two local minima and two local maxima.*

Definition 1.5. A knot is a simple closed curve in $\mathbb{R}^3$.

Definition 1.6. The total absolute curvature of a knot K is the integral of the absolute value of the curvature, taken over the curve, i.e. it is the quantity

$$\oint_K |\kappa(s)|\, ds.$$

Example. The total absolute curvature of a circle is always 2π.

Theorem 1.2 (Fáry-Milnor Theorem)**.** *If the total absolute curvature of a knot K is at most 4π, then K is an unknot.*

Definition 1.7. An immersed loop γ is such that γ' is never zero.

Definition 1.8. Two loops are isotopic if there exists an interpolating family of loops between them. Two immersed loops are isotopic if we can choose such an interpolating family of immersed loops.

Example. Without the restriction of immersion, any two loops $\gamma, \eta\colon S^1 \to \mathbb{R}^n$ would be isotopic, since we can always construct the linear interpolations

$$H\colon S^1 \times [0,1] \to \mathbb{R}^n, \qquad H(e^{i\theta}, t) = (1-t)\gamma(e^{i\theta}) + t\eta(e^{i\theta}).$$

Theorem 1.3 (Hirsch-Smale Theory).

1. *Any two immersed loops in $\mathbb{R}^2$ are isotopic if and only if their turning numbers match.*
2. *Any two immersed loops in S^2 are isotopic if and only if their turning numbers modulo 2 match.*

1.2 Whitney's theorem

Lemma 1.4. *Let $\Omega \subset \mathbb{R}^n$ be open and let $C \subseteq \Omega$ be closed. Then there exists a continuous function $f \colon \Omega \to \mathbb{R}$ such that $f^{-1}(0) = C$.*

Remark. The converse, i.e. $f^{-1}(0) = C$ implies C is closed, where f is continuous on Ω, is trivial.

Proof. Set f to be the distance function from C, i.e.

$$f(x) = \inf_{y \in C} d(x, y).$$

□

Theorem 1.5 (Whitney's Theorem). *Let $\Omega \subset \mathbb{R}^n$ be open and let $C \subseteq \Omega$ be closed. Then there exists a smooth function $f \colon \Omega \to \mathbb{R}$ such that $f^{-1}(0) = C$.*

Proof. Set $V = \Omega \setminus C$, and cover V by a countable collection of open balls,

$$V = \bigcup_{i=1}^{\infty} B(q_k, r_k).$$

This can always be done since V is open, and using the density of $\mathbb{Q}$ in $\mathbb{R}$ to pick only rational q_k, r_k. Now for each open ball $B(q_k, r_k)$, we can construct a smooth bump functions f_k such that $f^{-1}(0) = \mathbb{R}^n \setminus B(q_k, r_k)$, $f^{-1}(1) = \overline{B(q_k, r_k/2)}$, and all derivatives of f_k vanish on $\mathbb{R}^n \setminus B(q_k, r_k)$.

Define the weights

$$c_k = \max_{\substack{|\alpha| \leq k \\ y \in \overline{B(q_k, r_k)}}} \left| \frac{\partial^\alpha f_k}{\partial x^\alpha}(y) \right|.$$

Note that each c_k is well-defined: there are finitely many multi-indices α given k, and each of the partials $\partial^\alpha f / \partial x^\alpha$ is a smooth function over a compact set, hence bounded. Furthermore, each $c_k \geq 1$. Finally, set

$$f = \sum_{k=1}^{\infty} \frac{f_k}{2^k c_k}.$$

It is clear that $f^{-1}(0) = C$. We can show that the partial sums s_n converge; let $\epsilon > 0$ and choose sufficiently large N such that $1/2^N < \epsilon$. Now for $m > n \geq N$, examine

$$|s_m(x) - s_n(x)| = \sum_{k=n+1}^{m} \frac{f_k(x)}{2^k c_k} \leq \sum_{k=n+1}^{m} \frac{1}{2^k} \leq \frac{1}{2^N} < \epsilon$$

Thus, the convergence is uniform, and f is C^0. For higher derivatives, we examine some α partial of the sums, and use the same argument; at each stage, $|\partial^\alpha f / \partial x^\alpha| < c_k$ whenever $|\alpha| < k$. □

1.3 Parametrized curves

Definition 1.9. A parametrized curve in $\mathbb{R}^n$ is a smooth map $\gamma\colon (\alpha, \beta) \to \mathbb{R}^n$ for some α, β with $-\infty \leq \alpha < \beta \leq \infty$.

Remark. Here, we will always implicitly assume that maps are continuous.

Remark. Such a curve is called regular if $\gamma'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Example. The curve defined by

$$\gamma\colon \mathbb{R} \to \mathbb{R}^n, \qquad t \mapsto a + tb$$

is a straight line through the point a, in the direction b.

Example. The curve defined by

$$\gamma\colon \mathbb{R} \to \mathbb{R}^2, \qquad t \mapsto (\cos t, \sin t)$$

is the unit circle in $\mathbb{R}^2$, counter-clockwise.

Example. The curve defined by

$$\gamma\colon \mathbb{R} \to \mathbb{R}^3, \qquad t \mapsto (t, \cos t, \sin t)$$

is a helix in $\mathbb{R}^3$, wrapped around the x-axis.

Definition 1.10. A diffeomorphism is a smooth map with a smooth inverse.

Example. Suppose that $\gamma\colon (\alpha, \beta) \to \mathbb{R}^n$ is a smooth curve. If we have a diffeomorphism $\varphi\colon (\alpha', \beta') \to (\alpha, \beta)$, then the smooth curve $\eta = \gamma \circ \varphi$ is a reparametrization of γ. Note that

$$\eta'(t) = \gamma'(\varphi(t))\, \varphi'(t).$$

Lemma 1.6. *If $\varphi\colon (\alpha', \beta') \to (\alpha, \beta)$ is a diffeomorphism, then $\varphi'(t) \neq 0$ for all $t \in (\alpha', \beta')$.*

Definition 1.11. If the diffeomorphism $\varphi' > 0$, we say that it it orientation preserving. If $\varphi' < 0$, we say that it is orientation reversing.

Definition 1.12. The arc length of a differentiable curve $\gamma\colon (\alpha, \beta) \to \mathbb{R}^n$, starting at t_0, is defined as

$$s(t) = \int_{t_0}^{t} \|\gamma'(u)\| \, du.$$

We call s the arc length parameter.

Remark. If $\gamma'(t) \neq 0$, then $s'(t) > 0$.

Definition 1.13. A unit speed curve γ is one where $\|\gamma'\| = 1$

Lemma 1.7. *Let γ be a regular smooth curve. Then its arc length parameter is a smooth function.*

Proof. Note that γ' and $\langle \cdot, \cdot \rangle$ are smooth functions. Thus,

$$\frac{ds}{dt} = \|\gamma'(t)\| = \sqrt{\langle \gamma'(t), \gamma'(t) \rangle} > 0,$$

showing that ds/dt is smooth. □

Lemma 1.8. *The arc length function s is a diffeomorphism onto its image.*

Proof. This follows from the Inverse Function Theorem, using the smoothness of s. □

Lemma 1.9. *Let φ denote $s^{-1}\colon (\alpha', \beta') \to (\alpha, \beta)$. Then, $\gamma \circ \varphi$ is a unit speed reparametrization of γ.*

Remark. Any other unit speed reparametrization is related to s by shifts and reflections.

Proof. Note that s is strictly increasing, so $s', \varphi' > 0$. Now,

$$\|(\gamma \circ \varphi)'(t)\| = \|\gamma'(\varphi(t))\| \cdot |\varphi'(t)| = s'(\varphi(t))\varphi'(t) = (s \circ \varphi)'(t) = 1.$$

□

1.4 Curvature

Definition 1.14. Let $\gamma\colon (\alpha, \beta) \to \mathbb{R}^n$ be a regular curve. Let Δs be the length of the curve from $\gamma(t)$ to $\gamma(t + \Delta t)$, and let $\Delta\theta$ be the angle between these two vectors. Then, the curvature of γ at $\gamma(t)$ is defined as

$$\lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta s}.$$

Remark. For a unit speed curve, the curvature is precisely $\|\gamma''(s)\|$.

Example. For a straight line $a + bt$, the curvature vanishes identically.

Example. For a circle of radius R, the curvature is $1/R$. Note that we parametrize

$$\gamma(s) = (x_0 + R\cos(t/R), y_0 + R\sin(t/R)).$$

Definition 1.15. Let $\gamma\colon (\alpha, \beta) \to \mathbb{R}^3$ be a regular C^2 curve. Its curvature is defined as

$$\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}.$$

Remark. It is easy to check that the curvature at a point is independent of parametrization.

Definition 1.16. Given a C^2 plane curve γ such that $\ddot{\gamma}(0) \neq 0$, it is said to turn to the right when $\det(\dot{\gamma}(0), \ddot{\gamma}(0))$ is negative.

Definition 1.17. Let $\gamma\colon (\alpha, \beta) \to \mathbb{R}^n$ be a regular C^2 curve. Its curvature is defined as

$$\kappa(t) = \frac{\|\gamma''\langle\gamma', \gamma'\rangle - \gamma'\langle\gamma', \gamma''\rangle\|}{\|\gamma'(t)\|^4}.$$

Definition 1.18. Consider a regular smooth curve γ, such that $\ddot{\gamma}(s) \neq 0$ at s. Then, $\dot{\gamma}(s)$ and $\ddot{\gamma}(s)$ are perpendicular, and span the osculating plane at $\gamma(s)$.

Theorem 1.10. *Consider a regular smooth curve $\gamma\colon (\alpha, \beta) \to \mathbb{R}^n$, such that $\ddot{\gamma}(s) \neq 0$ at s.*

1. *For s_1, s_2, s_3 sufficiently close to s, the points $\gamma(s_i)$ are not colinear.*
2. *As s_1, s_2, s_3 tend to s, the planes $A(s_1, s_2, s_3)$ tend to the osculating plane at $\gamma(s)$.*
3. *The circumcircle $C(s_1, s_2, s_3)$ associated with these points tend to a circle $C(s)$ lying in the osculating plane which passes through $\gamma(s)$. Furthermore, this has radius $1/\|\ddot{\gamma}(s)\|$.*
4. *For s_1 sufficiently close to s, there is a unique plane containing $\gamma(s_1)$ and the tangent line to γ at s. As s_1 tends to s, these planes converge to the osculating plane at $\gamma(s)$.*

Proof. Let $\{(s_{1n}, s_{2n}, s_{3n})\}_{n=1}^{\infty}$ be a sequence converging to (s, s, s); assume that $s_{1n} < s_{2n} < s_{3n}$. Suppose that $\gamma(s_{1n})$, $\gamma(s_{2n})$, $\gamma(s_{3n})$ line on a line ℓ_n, for every n. Define the planes $V_n = \ell_n^{\perp}$, and look at the functions

$$f_n^v(t) = \langle\gamma(t) - \gamma(s_{1n}), v\rangle, \qquad v \in V_n.$$

Notice that s_{1n}, s_{2n}, s_{3n} are zeroes of f_n^v. Thus, we can choose s_{12n}, s_{23n}, where $s_{1n} \neq s_{12n} \leq s_{2n} \leq s_{23n} \leq s_{3n}$, such that $(f_n^v)'(s_{12n}) = (f_n^v)'(s_{23n}) = 0$. This gives

$$\langle \gamma'(s_{12n}), v\rangle = \langle \gamma'(s_{23n}), v\rangle = 0.$$

Repeating yields a point s_n such that $\langle \gamma''(s_n), v\rangle = 0$. Now, there is a neighbourhood of s on which

$$\|\gamma'(u) - \gamma'(s)\| < \epsilon, \qquad \|\gamma''(u) - \gamma''(s)\| < \epsilon.$$

As a result,

$$\langle \gamma'(s), v\rangle \leq \|v\|\epsilon, \qquad \langle \gamma''(s), v\rangle \leq \|v\|\epsilon.$$

Thus, given a vector in the osculating plane, $w = a\gamma'(s) + b\gamma''(s)$, we have $\|w\|^2 = a^2 + b^2k^2$, and

$$|\langle w, v\rangle| \leq (|a| + |b|)\|v\|\epsilon \leq c\|w\|\|v\|\epsilon.$$

This means that the osculating plane is part of the ϵ-perpendicular region to V_n. □

Lemma 1.11. *Let d be the Euclidean distance between $\gamma(0)$ and $\gamma(s)$, and s be the arc length between these two points. Then,*

$$\lim_{s\to 0} \frac{d}{s} = 1, \qquad \lim_{s\to 0} \frac{d-s}{s^3} = -\frac{1}{24}\|\ddot{\gamma}(0)\|^2$$

1.5 Torsion

Definition 1.19. Let $\gamma\colon (\alpha, \beta) \to \mathbb{R}^3$ be a C^2, regular space curve such that $\ddot{\gamma}(s) \neq 0$. The torsion of $\gamma(s)$ is defined as

$$\tau(s) = \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s},$$

where $\Delta\theta$ is the angle between the osculating planes to γ at s and $s + \Delta s$.

Remark. When we talk of the angle between two planes, we examine the normals

$$n(s) = \frac{\dot{\gamma}(s) \times \ddot{\gamma}(s)}{\|\dot{\gamma}(s) \times \ddot{\gamma}(s)\|}.$$

Lemma 1.12. *The torsion at $\gamma(s)$ can be expressed at*

$$\tau(s) = \frac{\|(\dot{\gamma}(s) \times \ddot{\gamma}(s)) \cdot \dddot{\gamma}(s)\|}{\|\ddot{\gamma}(s)\|^2}.$$

Lemma 1.13. *The torsion at $\gamma(s)$ can be expressed at*

$$\tau(t) = \frac{\det(\gamma'(t)\ \gamma''(t)\ \gamma'''(t))}{\|\gamma'(t) \times \gamma''(t)\|^2}$$

1.6 Frenet-Serret formulas

Definition 1.20. The tangent, principal normal, and binormal at $\gamma(s)$ are

$$t(s) = \dot{\gamma}(s), \qquad n(s) = \frac{\ddot{\gamma}(s)}{\|\ddot{\gamma}(s)\|}, \qquad b(s) = t(s) \times n(s).$$

Theorem 1.14 (Frenet-Serret)**.** *For a unit speed curve with nowhere vanishing curvature, the following holds.*

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

Corollary 1.14.1. *A regular space curve with non-zero curvature everywhere is planar if and only if $\tau = 0$.*

Corollary 1.14.2. *Under suitable conditions, a space curve is completely determined by κ, τ. Two unit speed curves having the same curvature and torsion functions are identical, up to a rotation and a shift.*

Example. Suppose that a unit speed space curve γ lies entirely on a sphere of radius r. Then we have

$$\begin{aligned}
\|\gamma - a\|^2 &= r^2, \\
2\dot{\gamma} \cdot (\gamma - a) &= 0, \\
t \cdot (\gamma - a) &= 0, \\
\dot{t} \cdot (\gamma - a) + t \cdot \dot{\gamma} &= 0, \\
\kappa n \cdot (\gamma - a) + 1 &= 0, \\
n \cdot (\gamma - a) &= -1/\kappa, \\
\dot{n} \cdot (\gamma - a) + n \cdot \dot{\gamma} &= \dot{\kappa}/\kappa^2, \\
-\kappa t \cdot (\gamma - a) + \tau b \cdot (\gamma - a) &= \dot{\kappa}/\kappa^2, \\
b \cdot (\gamma - a) &= \dot{\kappa}/\kappa^2\tau, \\
\dot{b} \cdot (\gamma - a) + b \cdot \dot{\gamma} &= (\dot{\kappa}/\kappa^2\tau)', \\
-\tau n \cdot (\gamma - a) &= (\dot{\kappa}/\kappa^2\tau)', \\
\tau/\kappa &= (\dot{\kappa}/\kappa^2\tau)'.
\end{aligned}$$

Thus, our curve satisfies

$$\frac{\tau}{\kappa} = \frac{d}{ds}\left(\frac{\dot{\kappa}}{\kappa^2\tau}\right).$$

By setting $\rho = 1/\kappa$, $\sigma = 1/\tau$, this reads

$$\rho = -\sigma \frac{d}{ds}(\dot{\rho}\sigma).$$

Conversely, assume that the above holds. Consider the quantity $\rho^2 + (\dot{\rho}\sigma)^2$; differentiating this gives

$$2\rho\dot{\rho} + 2\dot{\rho}\sigma \frac{d}{ds}(\dot{\rho}\sigma) = 2\rho\dot{\rho} + 2\dot{\rho}(-\rho) = 0.$$

Thus, we have

$$\rho^2 + (\dot{\rho}\sigma)^2 = r^2$$

for some positive constant r. Now, define the curve

$$\alpha = \gamma + \rho n + \dot{\rho}\sigma b.$$

Then,

$$\dot{\alpha} = \dot{\gamma} + \dot{\rho}n + \rho\dot{n} + \frac{d}{ds}(\dot{\rho}\sigma)b + (\dot{\rho}\sigma)\dot{b} = t + \dot{\rho}n - \rho\kappa t + \rho\tau b - \rho/\sigma b - \dot{\rho}\sigma\tau n = 0.$$

This means that the curve α is constant, say $\alpha = a$, whence

$$\|\gamma - a\|^2 = \|\rho n + \dot{\rho}\sigma b\|^2 = \rho^2 + (\dot{\rho}\sigma)^2 = r^2,$$

hence γ lies on a sphere.

1.7 Evolutes and involutes

Definition 1.21. The evolute of a curve is the locus of the centres of its osculating circles.

Remark. Consider a curve γ, with normal n and radius of curvature $\rho = 1/\kappa$. Then the equation of its evolute is given by $\gamma + \rho n$.

Example. The evolute of a circle is a constant curve, namely its centre.

Example. The evolute of a cycloid is a shifted copy of itself.

Definition 1.22. The involute of a curve is the locus of a point on a piece of taut string, as the string is wrapped around the curve.

Remark. Consider a curve γ, with tangent t. Then the equations of its involutes are given by $\gamma - t(s - a)$, for different choices of a. All such involutes are merely shifted copies of themselves.

Theorem 1.15. *The evolute of an involute of a curve is the curve itself.*

2 Surfaces

2.1 Introduction

Definition 2.1. A surface $\Sigma \subseteq \mathbb{R}^3$ is a subset satisfying the property that for any $p \in \Sigma$, there exists an open set $W \subseteq \mathbb{R}^3$, an open set $U \subseteq \mathbb{R}^2$, and a homeomorphism $\varphi\colon U \to W \cap \Sigma$.

Remark. The pair $(W \cap \Sigma, \varphi^{-1})$ is called a chart around p.

Remark. Without loss of generality, we can demand that the open set $U \subseteq \mathbb{R}^2$ be the unit disc centred at 0.

Example. Any affine plane in $\mathbb{R}^3$ is a surface.

Example. The unit sphere S^2 is a surface. Note that the north hemisphere is homeomorphic to the unit disc via a projection map; this gives us a chart for $(0, 0, 1)$. By symmetry, we can produce similar charts for any point on the sphere.

Remark. The sphere cannot be realized as a surface using only a single chart. This is because S^2 is compact while the unit disc is not, hence they cannot be homeomorphic.

Example. The cylinder defined by $x^2 + y^2 = 1$ is a surface. Note that we can produce the homeomorphism $(1, \theta, z) \mapsto (e^z, \theta)$, which maps the cylinder to $\mathbb{R}^2 \setminus 0$.

Remark. Note that the cylinder is just $S^1 \times \mathbb{R}$, and the plane minus the origin is just $S^1 \times (0, \infty)$. Thus we need only find a homeomorphism mapping $\mathbb{R} \to (0, \infty)$.

Example. The following is a parametrization of a torus.

$$(u, v) \mapsto ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin v).$$

The torus is also the zero set of the map

$$(x, y, z) \mapsto (\sqrt{x^2 + y^2} - a)^2 + z^2 - b^2.$$

Example. Let $U \subseteq \mathbb{R}^2$ be open, and let $f\colon U \to \mathbb{R}$ be continuous. The graph of f, which is the set $\Gamma_f = \{(x, y, f(x, y)) : (x, y) \in U\}$, is a surface.

Lemma 2.1. *All homeomorphisms $\psi\colon \mathbb{R}^3 \to \mathbb{R}^3$ take surfaces to surfaces.*

Example. In particular, all rigid motions take surfaces to surfaces.

2.2 Smooth surfaces

Theorem 2.2. *Let a be a regular value of the smooth map $f\colon U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^3$ is open. Then, $S = f^{-1}(a)$ is a surface.*

Definition 2.2. A surface $\Sigma \subset \mathbb{R}^3$ is called smooth if it admits a covering by charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ such that $\varphi_\alpha^{-1}\colon \varphi_\alpha(U_\alpha) \to \mathbb{R}^3$ is smooth and $D_u(\varphi_\alpha^{-1})$, $D_v(\varphi_\alpha^{-1})$ are linearly independent everywhere.

Remark. Such charts are called regular.

Lemma 2.3. *Let Σ be a smooth surface. For every $p \in \Sigma$, there exists a smooth function $\varphi\colon U \to \mathbb{R}$ where $U \subseteq \mathbb{R}^2$ is open, such that Γ_φ defines a regular chart of Σ around p.*

2.3 Orientability

Definition 2.3. Consider two charts $\varphi\colon U \to M$, $\psi\colon V \to M$ such that $W = \varphi(U) \cap \psi(V)$ is non-empty. This means that we have two choices of applying coordinates on the intersection. The transition map on this patch is $\psi^{-1} \circ \varphi$, which is a homeomorphism between $\varphi^{-1}(W)$ and $\psi^{-1}(W)$.

Definition 2.4. An oriented atlas for a smooth surface Σ is one in which the determinants of the linearized transitions maps are all positive.

Remark. A surface which admits an oriented atlas is called an orientable surface.

Example. Spheres, tori, and their connected sums are all orientable.

Example. The Möbius strip and the Klein bottle are non-orientable.

Example. The real projective plane $\mathbb{R}P^2$ is non-orientable.

Definition 2.5. Let $\sigma\colon U \to \Sigma$ describe a chart on a patch $\sigma(U) \subseteq \Sigma$, where $U \subseteq \mathbb{R}^2$ is open. Further let $p \in \sigma(U)$ be a point in this patch on the surface. Then, the map σ describes a set of local coordinates at p, which in turn can be used to describe a normal to the surface at p as follows.

$$\mathbb{N}^\sigma(p) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}.$$

Lemma 2.4. *Consider two regular charts $\sigma, \tilde{\sigma}$ whose associated surface patches overlap. Then, the transition map $\varphi = \sigma^{-1} \circ \tilde{\sigma}$ is a diffeomorphism between the domains of $\sigma, \tilde{\sigma}$. If p is a point in this overlap, then the normals described by these charts to the surface at p are related as*

$$\mathbb{N}^{\tilde{\sigma}}(p) = \operatorname{sgn}(\det D\varphi)\, \mathbb{N}^{\sigma}(p).$$

Remark. The normal vector at p is orthogonal to the tangent space $T_p\Sigma$.

Remark. We have the relation

$$\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}} = \det(D\varphi)\, \sigma_u \times \sigma_v.$$

Definition 2.6. Let Σ be an oriented surface. Given a point $p \in \Sigma$, the normal vectors $\mathbb{N}^{\sigma}(p)$ all agree irrespective of our choice of chart. We can now define the Gauss map $\mathbb{N}\colon \Sigma \to S^2$ which assigns a normal vector to each point $p \in \Sigma$ in our surface.

Lemma 2.5. *The following are equivalent for surfaces $\Sigma \subset \mathbb{R}^3$.*

1. *There exists an oriented atlas for Σ.*
2. *There exists a unit normal vector field $\mathbb{N}\colon \Sigma \to S^2$ such that $\mathbb{N}(p) \perp T_p\Sigma$ at every point $p \in \Sigma$.*

Theorem 2.6. *Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a smooth map. If $a \in \mathbb{R}$ is a regular value of f, then $f^{-1}(a) = \Sigma_a$ is an orientable smooth surface.*

Remark. Given $p \in \Sigma_a$, it can be shown that $\nabla f \perp T_p\Sigma_a$, hence we have the unit normal vector field $\nabla f / \|\nabla f\|$.

2.4 The First and Second Fundamental Forms

Definition 2.7. The following symmetric, non-degenerate, bilinear form is called the First Fundamental Form.

$$\mathrm{I}_p\colon T_p\Sigma \times T_p\Sigma \to \mathbb{R}, \qquad (v, w) \mapsto \langle v, w \rangle.$$

Since σ_u, σ_v form a basis of $T_p\Sigma$, we set

$$E = \langle \sigma_u, \sigma_u \rangle, \qquad F = \langle \sigma_u, \sigma_v \rangle, \qquad G = \langle \sigma_v, \sigma_v \rangle.$$

and we denote

$$\mathrm{I}_p^{\sigma} \equiv \begin{bmatrix} E & F \\ F & G \end{bmatrix} \equiv E\, du^2 + 2F\, du\, dv + G\, dv^2.$$

Lemma 2.7. *Consider two regular charts $\sigma, \tilde{\sigma}$ whose associated surface patches overlap. Then, the transition map $\varphi = \sigma^{-1} \circ \tilde{\sigma}$ is a diffeomorphism between the domains of $\sigma, \tilde{\sigma}$. If p is a point in this overlap, then the values E, F, G and $\tilde{E}, \tilde{F}, \tilde{G}$ are related as*

$$\begin{bmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{bmatrix} = D\varphi^\top \begin{bmatrix} E & F \\ F & G \end{bmatrix} D\varphi.$$

Lemma 2.8.

$$\|\sigma_u \times \sigma_v\|^2 = \det \begin{bmatrix} E & F \\ F & G \end{bmatrix}.$$

Definition 2.8. Let (U, σ) be a regular chart, and let $R \subseteq U$ be open. The area of the region $\sigma(R)$ is defined as

$$A_\sigma(R) = \iint_R \|\sigma_u \times \sigma_v\| \, du \, dv.$$

Definition 2.9. A regular chart (U, σ) is called conformal if $E \equiv G$ and $F \equiv 0$. Thus, the First Fundamental Form looks like

$$\mathrm{I}_p^\sigma = f(u, v)(du^2 + dv^2),$$

for some positive function f. The resultant coordinates u, v are called isothermal.

Remark. Conformal charts are angle preserving.

Definition 2.10. Let (U, σ) be a regular chart. Set

$$\begin{aligned} L &= \sigma_{uu} \cdot \mathbb{N}^\sigma = -\sigma_u \cdot \mathbb{N}^\sigma_u, \\ M &= \sigma_{uv} \cdot \mathbb{N}^\sigma = -\sigma_u \cdot \mathbb{N}^\sigma_v = -\sigma_v \cdot \mathbb{N}^\sigma_u, \\ N &= \sigma_{vv} \cdot \mathbb{N}^\sigma = -\sigma_v \cdot \mathbb{N}^\sigma_v. \end{aligned}$$

The Second Fundamental Form is denoted

$$\mathrm{II}_p^\sigma \equiv \begin{bmatrix} L & M \\ M & N \end{bmatrix} \equiv L \, du^2 + 2M \, du \, dv + N \, dv^2.$$

Lemma 2.9. *Let $\varphi = \sigma^{-1} \circ \tilde{\sigma}$ be a transition map between two regular charts. Set*

$$J = D\varphi = \begin{bmatrix} \partial u/\partial \tilde{u} & \partial u/\partial \tilde{v} \\ \partial v/\partial \tilde{u} & \partial v/\partial \tilde{v} \end{bmatrix}.$$

Then, the values L, M, N, and $\tilde{L}, \tilde{M}, \tilde{N}$ are related as

$$\begin{bmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{N} \end{bmatrix} = \operatorname{sgn}(\det J)\, J^\top \begin{bmatrix} L & M \\ M & N \end{bmatrix} J.$$

Example. Consider the following parametrization of a sphere.

$$\sigma\colon (-\pi/2, \pi/2) \times (0, 2\pi) \to \mathbb{R}^3, \qquad (u, v) \mapsto (R\cos u \cos v, R\cos u \sin v, R\sin u).$$

We can compute

$$\begin{aligned}
\sigma_u(u, v) &= (-R\sin u \cos v, -R\sin u \sin v, R\cos u), \\
\sigma_v(u, v) &= (-R\cos u \sin v, R\cos u \cos v, 0), \\
\sigma_u \times \sigma_v &= (-R^2\cos^2 u \cos v, -R^2\cos^2 u \sin v, -R^2\cos u \sin u), \\
\|\sigma_u \times \sigma_v\| &= R^2 \cos u, \\
\mathbb{N}^\sigma(u, v) &= (-\cos u \cos v, -\cos u \sin v, -\sin u) = -\sigma/R, \\
\mathbb{N}^\sigma_u(u, v) &= (\sin u \cos v, \sin u \sin v, -\cos u) \\
\mathbb{N}^\sigma_v(u, v) &= (\cos u \sin v, -\cos u \cos v, 0).
\end{aligned}$$

Thus,

$$E = R^2, \qquad F = 0, \qquad G = R^2\cos^2 u,$$
$$L = R, \qquad M = 0, \qquad N = R\cos^2 u.$$

The First and Second Fundamental Forms are

$$\mathrm{I}^\sigma = R^2(du^2 + \cos^2 u\, dv^2), \qquad \mathrm{II}^\sigma = R(du^2 + \cos^2 u\, dv^2).$$

Example. Consider the following parametrization of a torus.

$$\sigma\colon (0, 2\pi) \times (0, 2\pi) \to \mathbb{R}^3, \qquad (u, v) \mapsto ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u).$$

We can compute

$$\begin{aligned}
\sigma_u(u, v) &= (-b\sin u \cos v, -b\sin u \sin v, b\cos u), \\
\sigma_v(u, v) &= (-(a + b\cos u)\sin v, (a + b\cos u)\cos v, 0), \\
\sigma_u \times \sigma_v &= (-b(a + b\cos u)\cos u \cos v, -b(a + b\cos u)\cos u \sin v, -b(a + b\cos u)\sin u), \\
\|\sigma_u \times \sigma_v\| &= b(a + b\cos u) \\
\mathbb{N}^\sigma(u, v) &= (-\cos u \cos v, -\cos u \sin v, -\sin u) \\
\mathbb{N}^\sigma_u(u, v) &= (\sin u \cos v, \sin u \sin v, -\cos u) = -\frac{1}{b}\sigma_u \\
\mathbb{N}^\sigma_v(u, v) &= (\cos u \sin v, -\cos u \cos v, 0) = -\frac{\cos u}{a + b\cos u}\sigma_v.
\end{aligned}$$

Thus,

$$E = b^2, \qquad F = 0, \qquad G = (a + b\cos u)^2,$$
$$L = b, \qquad M = 0, \qquad N = (a + b\cos u)\cos u.$$

The First and Second Fundamental Forms are

$$\mathrm{I}^\sigma = b^2\, du^2 + (a + b\cos u)^2\, dv^2, \qquad \mathrm{II}^\sigma = b\, du^2 + (a + b\cos u)\cos u\, dv^2.$$

2.5 Curvature and the Shape Operator

Definition 2.11. The shape operator $\mathscr{S}_p \colon T_p\Sigma \to T_p\Sigma$ is defined as follows. Given $v \in T_p\Sigma$, pick a representative curve $\gamma \colon (-\epsilon, \epsilon) \to \Sigma$ so that $v = [\gamma]$. Next, set $\eta = \sigma^{-1} \circ \gamma$, and compute $w = -(d/dt)(\mathbb{N}^\sigma \circ \eta)\big|_{t=0} \in T_{\sigma^{-1}(p)}U$. Finally, identify w with its corresponding element in $T_p\Sigma$.

Remark. We have $\mathscr{S}_p \equiv -D\mathbb{N}^\sigma_p$.

Lemma 2.10. *The shape operator is self-adjoint.*

Proof. We claim that

$$\langle \mathscr{S}_p v, w\rangle = \langle v, \mathscr{S}_p w\rangle$$

for all $v, w \in T_p\Sigma$. It is sufficient to check this for the basis vectors σ_u, σ_v. Now,

$$\langle \mathscr{S}_p \sigma_u, \sigma_u\rangle = \langle -D_{\partial u}\mathbb{N}^\sigma\big|_p, \sigma_u\rangle = -\langle \mathbb{N}^\sigma_u, \sigma_u\rangle = L.$$

It can be similarly checked that

$$\langle \mathscr{S}_p \sigma_u, \sigma_v\rangle = \langle \mathscr{S}_p \sigma_v, \sigma_u\rangle = M, \qquad \langle \mathscr{S}_p \sigma_v, \Sigma_v\rangle = N.$$ □

Lemma 2.11. *The matrix of $\mathscr{S}^\sigma_p$ in the basis $\{\sigma_u, \sigma_v\}$ is given by*

$$\mathscr{S}^\sigma_p = \mathrm{I}_p^{-1}\mathrm{II}_p = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} L & M \\ M & N \end{bmatrix}.$$

Furthermore, let $\varphi = \sigma^{-1} \circ \tilde{\sigma}$ be a transition map between two regular charts, and set $J = D\varphi$. Then,

$$\mathscr{S}^{\tilde{\sigma}}_p = \pm J^{-1}\mathscr{S}^\sigma_p J.$$

Corollary 2.11.1. *The product of the eigenvalues of $\mathscr{S}_p$ is invariant under coordinate transformations, and the sum of the eigenvalues changes by at most a sign.*

Definition 2.12. Let k_1, k_2 be the two eigenvalues of $\mathscr{S}_p^\sigma$. These are called the principal curvatures of at p. We further define the Gaussian curvature

$$K = k_1 k_2 = \det \mathscr{S}_p^\sigma$$

and the mean curvature

$$H = \frac{1}{2}(k_1 + k_2) = \frac{1}{2} \operatorname{trace} \mathscr{S}_p^\sigma.$$

Remark. This immediately gives

$$K = \frac{LN - M^2}{EG - F^2}.$$

Lemma 2.12.

$$\mathbb{N}_u^\sigma \times \mathbb{N}_v^\sigma = K\, \sigma_u \times \sigma_v.$$

Proof. Note that σ_u, σ_v forms a basis of $T_p\Sigma$, hence we can expand

$$\mathbb{N}_u^\sigma = a\sigma_u + b\sigma_v, \qquad \mathbb{N}_v^\sigma = c\sigma_u + d\sigma_v.$$

Now,

$$\begin{aligned} -L &= \mathbb{N}_u^\sigma \cdot \sigma_u = aE + bF, \\ -M &= \mathbb{N}_u^\sigma \cdot \sigma_v = aF + bG, \\ -M &= \mathbb{N}_v^\sigma \cdot \sigma_u = cE + dF, \\ -N &= \mathbb{N}_u^\sigma \cdot \sigma_u = cF + dG. \end{aligned}$$

Then,

$$\begin{bmatrix} -L & -M \\ -M & -N \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \qquad \begin{bmatrix} a & c \\ b & d \end{bmatrix} = -\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} L & M \\ M & N \end{bmatrix}$$

Thus,

$$\mathbb{N}_u^\sigma \times \mathbb{N}_v^\sigma = (ad - bc)\, \sigma_u \times \sigma_v = \det(\mathscr{S}_p^\sigma)\, \sigma_u \times \sigma_v = K\, \sigma_u \times \sigma_v.$$

□

Corollary 2.12.1. *Let the Gauss map $\mathbb{N}$ map $U \subseteq \mathbb{R}^2$ bijectively to $\mathbb{N}(U) \subseteq S^2$ bijectively. Then,*

$$\iint_{\sigma(U)} |K|\, dA = A(\mathbb{N}(U)).$$

Definition 2.13. A surface Σ is called minimal if $H \equiv 0$.

Remark. Note that $H = 0$ forces $K \leq 0$.

Theorem 2.13. *If Σ is a compact surface, then there exists a point $p \in \Sigma$ such that $K(p) > 0$.*

Proof. Consider the map

$$f : \Sigma \to \mathbb{R}, \qquad v \mapsto \|v\|^2.$$

This map must attain its maximum and minimum at points $p^+, p^- \in \Sigma$. At such points, $Df_p \equiv 0$, but

$$Df_p : T_p\Sigma \to T_{f(p)}\mathbb{R}, \qquad v \mapsto 2\langle p, v\rangle.$$

This shows that $p \perp T_p\Sigma$, hence $\mathbb{N}_p^\sigma = \pm\hat{p}$ when p is such an extremal point. Consider a regular curve $\gamma \in T_{p^+}\Sigma$; note that $f \circ \gamma$ attains its maximum at 0, hence

$$\frac{d}{dt} f(\gamma(t))\Big|_{t=0} = 0, \qquad \frac{d^2}{dt^2} f(\gamma(t))\Big|_{t=0} \leq 0.$$

Thus,

$$2\langle\gamma(0), \dot{\gamma}(0)\rangle = 0, \qquad 2[\|\dot{\gamma}(0)\|^2 + \langle\gamma(0), \ddot{\gamma}(0)\rangle] \leq 0.$$

Now $\|\dot{\gamma}\| = 1$, $\gamma(0) = p^+$, and $\|\ddot{\gamma}(0)\|$ is the curvature κ, so $\kappa \geq -1/\|p^+\|$. Hence, the principal curvatures at p^+ also satisfy this constraint giving

$$K(p^+) = k_1 k_2 \geq \frac{1}{\|p^+\|^2} > 0.$$

The same can be said of $K(p^-)$; note that $\|p\| > 0$ at these points without loss of generality, as we can always shift the surface Σ so as not to enclose the origin. □

Corollary 2.13.1. *There are no closed (compact, without boundary) minimal surfaces in $\mathbb{R}^3$.*

Theorem 2.14 (Liebmann)**.** *If Σ is a connected smooth compact surface of constant Gaussian curvature K, then $K > 0$ and Σ is a sphere of radius $1/\sqrt{K}$.*

Theorem 2.15 (Hopf)**.** *If Σ is a smooth surface homeomorphic to a sphere and with constant mean curvature H, then Σ is a sphere of radius $1/|H|$.*

Theorem 2.16. *Let k_1, k_2 be the principal curvatures of Σ for a regular surface patch (U, σ). Assume that $|k_1|, |k_2| < c$ on $\sigma(U)$, and let λ be a constant such that $\lambda < 1/c$. Set*

$$\sigma^\lambda = \sigma + \lambda \mathbb{N}^\sigma.$$

Then, the following hold.

1. *σ^λ also describes a regular patch.*
2. *$\mathbb{N}^{\sigma^\lambda} = \mathbb{N}^\sigma$.*
3. $$k_1(\sigma^\lambda) = \frac{k_1}{1 - \lambda k_1}, \qquad k_2(\sigma^\lambda) = \frac{k_2}{1 - \lambda k_2}.$$
4. $$K(\sigma^\lambda) = \frac{K}{1 - 2\lambda H + \lambda^2 K}, \qquad H(\sigma^\lambda) = \frac{H - \lambda K}{1 - 2\lambda H + \lambda^2 K}.$$
5. $$A(\Sigma^\lambda) = A(\Sigma) - 2\lambda \int_\Sigma H \, dA + \lambda^2 \int_\Sigma K \, dA.$$

2.6 Gauss's Theorema Egregium

Theorem 2.17. *Let (U, σ) be a regular surface patch, and further let it be an orthogonal patch, i.e. $F \equiv 0$. Then, the Gaussian curvature is given by*

$$K = -\frac{1}{2\sqrt{EG}} \left[\left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right].$$

Remark. If we further set $E \equiv G \equiv e^{2\rho}$, then

$$K = -e^{-2\rho}(\rho_{uu} + \rho_{vv}).$$

Proof. Fix a point $p \in U$, and note that the vectors $\sigma_u, \sigma_v, \mathbb{N}^\sigma$ form an orthogonal basis. Thus, we can expand

$$\begin{aligned} \sigma_{uu} &= \Gamma^u_{uu}\sigma_u + \Gamma^v_{uu}\sigma_v + L\mathbb{N}^\sigma, \\ \sigma_{uv} &= \Gamma^u_{uv}\sigma_u + \Gamma^v_{uv}\sigma_v + M\mathbb{N}^\sigma, \\ \sigma_{vv} &= \Gamma^u_{vv}\sigma_u + \Gamma^v_{vv}\sigma_v + N\mathbb{N}^\sigma, \end{aligned}$$

and

$$\begin{aligned} \mathbb{N}^\sigma_u &= -\frac{L}{E}\sigma_u - \frac{M}{G}\sigma_v, \\ \mathbb{N}^\sigma_v &= -\frac{M}{E}\sigma_u - \frac{N}{G}\sigma_v. \end{aligned}$$

The functions $\Gamma^\gamma_{\alpha\beta}$ are called *Christoffel symbols.* Now,

$$\begin{aligned}
E_u &= \frac{\partial}{\partial u}(\sigma_u \cdot \sigma_u) = 2\sigma_u \cdot \sigma_{uu} = 2\Gamma^u_{uu}E, & \Gamma^u_{uu} &= \frac{E_u}{2E}, \\
E_v &= \frac{\partial}{\partial v}(\sigma_u \cdot \sigma_u) = 2\sigma_u \cdot \sigma_{uv} = 2\Gamma^u_{uv}E, & \Gamma^u_{uv} &= \frac{E_v}{2E}, \\
G_u &= \frac{\partial}{\partial u}(\sigma_v \cdot \sigma_v) = 2\sigma_v \cdot \sigma_{uv} = 2\Gamma^v_{uv}G, & \Gamma^v_{uv} &= \frac{G_u}{2G}, \\
G_v &= \frac{\partial}{\partial v}(\sigma_v \cdot \sigma_v) = 2\sigma_v \cdot \sigma_{vv} = 2\Gamma^v_{vv}G, & \Gamma^v_{vv} &= \frac{G_v}{2G}.
\end{aligned}$$

Finally, note that $\sigma_u \cdot \sigma_v = 0$, hence differentiating gives

$$\begin{aligned}
0 &= \sigma_v \cdot \sigma_{uu} + \sigma_u \cdot \sigma_{uv} = \Gamma^v_{uu}G + \Gamma^u_{uv}E, & \Gamma^v_{uu} &= -\frac{E_v}{2G}, \\
0 &= \sigma_u \cdot \sigma_{vv} + \sigma_v \cdot \sigma_{uv} = \Gamma^u_{vv}E + \Gamma^u_{uv}G, & \Gamma^v_{uu} &= -\frac{G_u}{2E}.
\end{aligned}$$

Now, we use Clairaut's theorem to equate the mixed partials $\sigma_{uuv} = \sigma_{uvu}$. Computing,

$$\begin{aligned}
\sigma_{uuv} &= (\Gamma^u_{uu})_v\sigma_u + \Gamma^u_{uu}\sigma_{uv} + (\Gamma^v_{uu})_v\sigma_v + \Gamma^v_{uu}\sigma_{vv} + L_v\mathbb{N}^\sigma + L\mathbb{N}^\sigma_v \\
&= \left[(\Gamma^u_{uu})_v + \Gamma^u_{uu}\Gamma^u_{uv} + \Gamma^v_{uu}\Gamma^u_{vv} - \frac{LM}{E}\right]\sigma_u + \left[\Gamma^u_{uu}\Gamma^v_{uv} + (\Gamma^v_{uu})_v + \Gamma^v_{uu}\Gamma^v_{vv} - \frac{LN}{G}\right]\sigma_v \\
&\quad + \left[\Gamma^u_{uu}M + \Gamma^v_{uu}N + L_v\right]\mathbb{N}^\sigma.
\end{aligned}$$

$$\begin{aligned}
\sigma_{uvu} &= (\Gamma^u_{uv})_u\sigma_u + \Gamma^u_{uv}\sigma_{uu} + (\Gamma^v_{uv})_u\sigma_v + \Gamma^v_{uv}\sigma_{uv} + M_u\mathbb{N}^\sigma + M\mathbb{N}^\sigma_u \\
&= \left[(\Gamma^u_{uv})_u + \Gamma^u_{uv}\Gamma^u_{uu} + \Gamma^v_{uv}\Gamma^u_{uv} - \frac{LM}{E}\right]\sigma_u + \left[\Gamma^u_{uv}\Gamma^v_{uu} + (\Gamma^v_{uv})_u + (\Gamma^v_{uv})^2 - \frac{M^2}{G}\right]\sigma_v \\
&\quad + \left[\Gamma^u_{uv}L + \Gamma^v_{uv}M + M_u\right]\mathbb{N}^\sigma.
\end{aligned}$$

Equating the coefficients of σ_v, we have

$$\frac{LN - M^2}{G} = (\Gamma^v_{uu})_v - (\Gamma^v_{uv})_u + \Gamma^u_{uu}\Gamma^v_{uv} + \Gamma^v_{uu}\Gamma^v_{vv} - \Gamma^u_{uv}\Gamma^v_{uu} - (\Gamma^v_{uv})^2.$$

Identifying that the Gaussian curvature $K = (LN - M^2)/EG$, we have shown that

$$EK = (\Gamma^v_{uu})_v - (\Gamma^v_{uv})_u + \Gamma^u_{uu}\Gamma^v_{uv} + \Gamma^v_{uu}\Gamma^v_{vv} - \Gamma^u_{uv}\Gamma^v_{uu} - (\Gamma^v_{uv})^2.$$

Expanding,

$$\begin{aligned}
EK &= -\left(\frac{E_v}{2G}\right)_v - \left(\frac{G_u}{2G}\right)_u + \frac{E_uG_u}{4EG} - \frac{E_vG_v}{4G^2} + \frac{E_v^2}{4EG} - \frac{G_u^2}{4G^2} \\
&= \frac{E_vG_v - E_{vv}G}{2G^2} + \frac{G_u^2 - G_{uu}G}{2G^2} + \frac{E_uG_u}{4EG} - \frac{E_vG_v}{4G^2} + \frac{E_v^2}{4EG} - \frac{G_u^2}{4G^2} \\
&= \frac{-E_{vv} - G_{uu}}{2G} + \frac{E_vG_v + G_u^2}{4G^2} + \frac{E_uG_u + E_v^2}{4EG}.
\end{aligned}$$

But

$$\begin{aligned}
\left(\frac{E_v}{\sqrt{EG}}\right)_v &= \frac{E_{vv}\sqrt{EG} - E_v(E_vG + EG_v)/2\sqrt{EG}}{EG}, \\
\left(\frac{G_u}{\sqrt{EG}}\right)_u &= \frac{G_{uu}\sqrt{EG} - G_u(E_uG + EG_u)/2\sqrt{EG}}{EG}.
\end{aligned}$$

Thus,

$$\begin{aligned}
-\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v+\left(\frac{G_u}{\sqrt{EG}}\right)_u\right] &= \frac{-E_{vv}-G_{uu}}{2EG}+\frac{E_v^2G+E_vG_vE+E_uG_uG+G_u^2E)}{4E^2G^2}\\
&= \frac{-E_{vv}-G_{uu}}{2EG}+\frac{E_vG_v+G_u^2}{4EG^2}+\frac{E_uG_u+E_v^2}{4E^2G}\\
&= K.
\end{aligned}$$

This proves the desired formula. □

Theorem 2.18 (Theorema Egregium)**.** *The Gaussian curvature K is determined by only the first fundamental form, i.e. the functions E, F, G, and their derivatives.*