MA3202

Algebra II

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Contents

1 Rings

1.1 Basic definitions

Definition 1.1. A ring is a set R equipped with two binary operations, namely addition and multiplication, such that

1. $(R, +)$ is an abelian group.
   (a) $a + b \in R$ for all $a, b \in R$.
   (b) $(a + b) + c = a + (b + c)$ for all $a, b, c \in R$.
   (c) $a + b = b + a$ for all $a, b \in R$.
   (d) There exists $0 \in R$ such that $a + 0 = a$ for all $a \in R$.
   (e) For each $a \in R$, there exists $-a \in R$ such that $a + (-a) = 0$.
2. $(R, \cdot)$ is a semi-group.
   (a) $a \cdot b \in R$ for all $a, b \in R$.
   (b) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in R$.
3. Multiplication distributes over addition.
   (a) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in R$.
   (b) $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all $a, b, c \in R$.

Remark. The following properties follow immediately,

1. $0 \cdot a = 0$ for all $a \in R$.
2. $(-a) \cdot b = -(a \cdot b) = a \cdot (-b)$ for all $a, b \in R$.
3. $(na) \cdot b = n(a \cdot b) = a \cdot (nb)$ for all $a, b \in R$.

Example. The integers $\mathbb{Z}$ form a ring, under the usual addition and multiplication.

Example. All fields, for instance the rational numbers $\mathbb{Q}$ or the real numbers $\mathbb{R}$, are rings.

Example. The integers modulo n, namely $\mathbb{Z}/n\mathbb{Z}$, form a ring.

Example. If R is a ring, then the algebra of polynomials $R[X]$ with coefficients from R form a ring.

Example. If R is a ring, then the $n \times n$ matrices $M_n(R)$ with entries from R form a ring.

Definition 1.2. If R is a ring and $(R, \cdot)$ is a monoid i.e. has an identity, then this identity is unique and called the unity of the ring R. Such a ring R is called a unit ring. Note that we typically demand that this identity be distinct from the zero element.

Example. The even integers $2\mathbb{Z}$ form a ring, but do not contain the identity.

Example. The trivial ring $\{0\}$ is typically not considered to be a unit ring, since 0 must serve as the additive identity as well as the multiplicative identity.

Definition 1.3. If R is a ring and $(R, \cdot)$ is commutative, then R is called a commutative ring.

Definition 1.4. Let R be a unit ring. An element $a \in R$ is called a unit if there exists $b \in R$ such that $a \cdot b = 1 = b \cdot a$. This $b \in R$ is unique, and denoted by a^{-1}.

Example. The units in $\mathbb{Z}$ are $\{1, -1\}$.

1.2 Subrings

Definition 1.5. Let R be a ring, and let $S \subseteq R$. We say S is a subring of R if the structure $(S, +, \cdot)$ is a ring, with addition and multiplication inherited from R.

Example. The rings $n\mathbb{Z}$ for $n \in \mathbb{N}$ are all subrings of $\mathbb{Z}$.

Example. Consider the rings $2\mathbb{Z} \subset \mathbb{Z}$. Here, $\mathbb{Z}$ is a unit ring but $2\mathbb{Z}$ is not.

Example. Consider the rings $4\mathbb{Z}/12\mathbb{Z} \subset 2\mathbb{Z}/12\mathbb{Z}$. Here, $2\mathbb{Z}/12\mathbb{Z}$ is not a unit ring but $4\mathbb{Z}/12\mathbb{Z}$ is.

Lemma 1.1. *Let S be a subring of R. Since $(R, +)$ is an abelian group, $(S, +)$ is a normal subgroup of $(R, +)$. Thus, we can make sense of the quotient group $(R/S, +)$.*

Lemma 1.2. *Let S be a subring of R. Then, the quotient $(R/S, +, \cdot)$ is a ring with multiplication $(a + S) \cdot (b + S) = ab + S$ if and only if $ab - xy \in S$ for all $a, b, x, y \in R$ such that the cosets $a + S = x + S$, $b + S = y + S$.*

Example. Consider the ring $\mathbb{Z}$ and the subring $n\mathbb{Z}$. Then, the quotient $\mathbb{Z}/n\mathbb{Z}$ is indeed a ring.

Example. Consider the ring $\mathbb{Q}$ and the subring $\mathbb{Z}$. It can be shown that $\mathbb{Q}/\mathbb{Z}$ is not a ring under the 'natural' multiplication.

1.3 Ideals

Definition 1.6. Let R be a ring and let I be a subset of R. We say that I is an ideal of R if $(I, +)$ is a subgroup of $(R, +)$, and $rx, xr \in I$ for all $r \in R$, $x \in I$.

Example. Consider the ring $\mathbb{Z}$, and the subring $n\mathbb{Z}$. This is an ideal of $\mathbb{Z}$, since $m(n\mathbb{Z}) \subseteq n\mathbb{Z}$. Indeed, every ideal of $\mathbb{Z}$ is of the form $n\mathbb{Z}$. This will follow from Euclid's Division Lemma.

Example. The subsets $\{0\}$ and R of any ring R are trivial ideals.

Lemma 1.3. *Let R be a ring, and I be an ideal of R. Then, the quotient R/I is a ring.*

Proof. Note that whenever $a - x \in I$, $b - y \in I$, we demand that $ab - xy \in I$. This can be rewritten as $(a - x)b + x(b - y) \in I$, which is clearly true by the properties of the ideal I. □

Definition 1.7. An ideal $I \subset R$ is called finitely generated if there exist $x_1, x_2, \ldots, x_n \in I$ such that every element of I can be written as a finite linear combination

$$x = r_1x_1 + \cdots + r_nx_n,$$

where $r_i \in R$. We denote $I = (x_1, x_2, \cdots, x_n)$.

Definition 1.8. An ideal generated by a single element is called a principal ideal.

Example. Every ideal of $\mathbb{Z}$ is a principal ideal.

Lemma 1.4. *Let R be a unit ring, and $I \subseteq R$ be an ideal. Then, $I = R$ if and only if I contains the identity.*

Definition 1.9. The sum of two ideals $I, J \subset R$ is defined

$$I + J = \{x + y : x \in I, y \in J\}.$$

Their product is defined

$$IJ = \{\sum_{i=1}^{n} x_iy_i : x_i \in I, y_i \in J\}.$$

Lemma 1.5. *The sum and product of two ideals of a ring are also ideals of that ring.*

Lemma 1.6. *Let $I, J \subset R$ be ideals in the commutative ring R. Then, $IJ \subset I \cap J$.*

Example. Note that for $2\mathbb{Z}, 2\mathbb{Z} \in \mathbb{Z}$, $(2\mathbb{Z})(2\mathbb{Z}) = 4\mathbb{Z}$ but $2\mathbb{Z} \cap 2\mathbb{Z} = 2\mathbb{Z}$. A related example is $R = 2\mathbb{Z}$, $I = 4\mathbb{Z}$, $J = 6\mathbb{Z}$.

Lemma 1.7. *If $I, J \subset R$ are ideals in a commutative unit ring R, and $I + J = R$, then $IJ = I \cap J$.*

Proof. We already know that $IJ \subseteq I + J$. Since $I + J = R$, we can pick $x \in I, y \in J$ such that $x + y = 1$. Now pick $a \in I \cap J$, hence $a \cdot 1 = ax + ay \in I \cap J$; but this is also an element of IJ proving $I \cap J \subseteq IJ$. □

1.4 Integral domains

Definition 1.10. Let R be a ring and $a, b \in R$, $a, b \neq 0$. If $ab = 0$, we call a a left zero divisor and b a right zero divisor.

Example. Consider $2, 3 \in \mathbb{Z}/6\mathbb{Z}$; then $2 \cdot 3 = 6 \equiv 0$.

Definition 1.11. A commutative ring R is called an integral domain if it has no zero divisors.

Example. When p is prime, the rings $\mathbb{Z}/p\mathbb{Z}$ are integral domains. Note that this set is a group under both $+$ and $\cdot$.

Lemma 1.8. *Every field is an integral domain.*

Theorem 1.9. *Every finite integral domain is a field.*

Proof. Let $R = \{x_1, \dots, x_n\}$ be a finite integral domain. We first show that R contains an identity 1. Pick $x \neq 0$, and note that $xx_1, xx_2, \dots, xx_n$ must all be distinct: otherwise $xx_i = xx_j$ would force $x(x_i - x_j) = 0$. This forces $x = xx_k$ for some $x_k \neq 0$. Now, we claim that x_k is our identity. Indeed, given any $y \neq 0$, we write $y = xx_l$ for some $x_l \neq 0$, hence $yx_k = xx_lx_k = x_l(xx_k) = x_lx = y$.

Next, we show that every non-zero $x \in R$ has an inverse. Indeed, $1 = x_k$ must be one of the $xx_1, \dots, xx_n$, hence $1 = xx_m$ for some non-zero x_m. This means that $x_m = x^{-1}$. □

Definition 1.12. Let R be a ring. The characteristic of R is the smallest positive integer n such that $nx = 0$ for all $x \in R$. If no such number n exists, we say that the characteristic of R is zero. We denote the characteristic of R by $\mathrm{ch}(R)$.

Example. We have $\mathrm{ch}(\mathbb{Z}) = 0$, $\mathrm{ch}(\mathbb{Z}/n\mathbb{Z}) = n$.

Lemma 1.10. *Let R be a unit ring. Then, $\mathrm{ch}(R)$ is the smallest positive integer n such that $n \cdot 1 = 0$; if no such n exists, then $\mathrm{ch}(R)$ is zero.*

Theorem 1.11. *Let R be an integral domain. Then, $\mathrm{ch}(R)$ is either zero or a prime.*

Proof. Let R be an integral domain such that $\mathrm{ch}(R) = n \neq 0$. If n is not a prime, write $n = n_1 n_2$ for $n_1, n_1 < n$. Then for any non-zero $x \in R$, write $0 = n(x^2) = (n_1 x)(n_2 x)$. This forces one of $n_1 x, n_2 x = 0$; say $n_1 x = 0$. Now for any $y \in R$, we have $x(n_1 y) = (n_1 x)y = 0$. Since $x \neq 0$, we have $n_1 y = 0$ for all $y \in R$, contradicting the minimality of n. □

1.5 Simple rings

Definition 1.13. A simple ring is one which has no non-trivial ideals. We typically demand that multiplication in R is non-trivial.

Lemma 1.12. *Every field is a simple ring.*

Proof. If R is a field and $I \subset R$ is an ideal with non-zero $a \in I$, then $a^{-1} \in R$ hence $a^{-1}a = 1 \in I$. This immediately forces $I = R$. □

Lemma 1.13. *If R is a commutative, simple, unit ring, then R is a field.*

Proof. Pick non-zero $a \in R$, and set $I = (a)$. Since R is simple, $I = R$, hence $1 \in I = (a)$. In other words, $1 = ab$ for some $b \in R$. □

1.6 Homomorphisms and isomorphisms

Definition 1.14. Let R, S be rings, and let $\varphi\colon R \to S$. We say that φ is a ring homomorphism if

1. $\varphi(x+y) = \varphi(x) + \varphi(y)$ for all $x, y \in R$.
2. $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in R$.
3. $\varphi(1_R) = 1_S$.

We only insist on 3 if both R and S are unit rings.

Remark. The following properties follow immediately.

1. $\varphi(0_R) = 0_S$.
2. $\varphi(-x) = -\varphi(x)$ for all $x \in R$.
3. $\varphi(nx) = n\varphi(x)$ for all $x \in R$, $n \in \mathbb{Z}$.
4. $\varphi(x-y) = \varphi(x) - \varphi(y)$ for all $x, y \in R$.

Example. The map $\varphi\colon \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, $k \mapsto k \bmod n$ is a homomorphism.

Definition 1.15. A bijective homomorphism between two rings is called an isomorphism. If an isomorphism exists between two rings, we say that they are isomorphic.

Example. The map $\varphi\colon \mathbb{Z} \to n\mathbb{Z}$, $k \mapsto nk$ is an isomorphism.

Example. The map $\varphi\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto \bar{z}$ is an isomorphism.

Example. The rings $\mathbb{Z}$ and $\mathbb{Q}$ are not isomorphic. If there did exist an isomorphism $\varphi\colon \mathbb{Q} \to \mathbb{Z}$, then set $a = \varphi(1/2)$. We now demand $a + a = \varphi(1/2 + 1/2) = 1$; but there is no such integer satisfying this property.

Lemma 1.14. *The only isomorphism $\mathbb{Z} \to \mathbb{Z}$ is the identity map.*

Theorem 1.15. *The only isomorphism $\mathbb{Q} \to \mathbb{Q}$ is the identity map.*

Proof. Let $\varphi\colon \mathbb{Q} \to \mathbb{Q}$ be an isomorphism. We must have $\varphi(1) = 1$, which immediately gives $\varphi(n) = n$ for all $n \in \mathbb{Z}$. Now for any rational $p/q \in \mathbb{Q}$, note that $1 = \varphi(q \cdot 1/q) = q \cdot \varphi(1/q)$, forcing $\varphi(1/q) = 1/q$. Thus, $\varphi(p/q) = p/q$, completing the proof. □

Theorem 1.16. *The only isomorphism $\mathbb{R} \to \mathbb{R}$ is the identity map.*

Proof. Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be an isomorphism. We must have $\varphi(q) = q$ for all $q \in \mathbb{Q}$.

First we show that φ is strictly increasing. Note that when $x > 0$, $\varphi(x) = \varphi(\sqrt{x})^2 > 0$. Thus when $x > y$, $\varphi(x-y) > 0$, hence $\varphi(x) > \varphi(y)$.

Now let $x \in \mathbb{R}$; if $\varphi(x) \neq x$, we must have one of $\varphi(x) > x$ or $\varphi(x) < x$. Assume the former, and find $q \in \mathbb{Q}$ such that $\varphi(x) > q > x$. Now, $q > x$ gives $q = \varphi(q) > \varphi(x)$, a contradiction. An analogous argument gives a contradiction when $\varphi(x) < x$, completing the proof. □

Theorem 1.17. *The only homomorphism $\mathbb{R} \to \mathbb{R}$ is the identity map.*

Proof. If $\varphi\colon \mathbb{R} \to \mathbb{R}$ is a homomorphism, it is easy to check that $\varphi^{-1}(0)$ is an ideal. Since $\mathbb{R}$ is simple, this must be $\{0\}$ or $\mathbb{R}$; the latter can be ruled out since $\varphi(1) = 1$. In other words, $\varphi^{-1} = \{0\}$ so φ is injective. Following the previous proof, φ must be an isomorphism, hence the identity map. □

Theorem 1.18. *The only isomorphisms $\mathbb{C} \to \mathbb{C}$ which send $\mathbb{R} \to \mathbb{R}$ are the maps $z \mapsto z$ and $z \mapsto \bar{z}$.*

Proof. The previous theorem guarantees that any such isomorphism $\varphi\colon \mathbb{C} \to \mathbb{C}$ is completely determined by $\varphi(i)$. Now, $-1 = \varphi(-1) = \varphi(i)^2$, forcing $\varphi(i) = \pm i$. □

Lemma 1.19. *The kernel of a ring homomorphism $\varphi\colon R \to S$ is an ideal of R. Its image is a subring of S.*

Proof. If $x \in \ker\varphi$, then $\varphi(x) = 0$, hence for any $r \in R$ we have $\varphi(rx) = \varphi(r)\varphi(x) = 0$. Thus, $rx \in \varphi^{-1}(0)$. Also, recall that $\varphi^{-1}(0)$ is an additive subgroup of R. □

Theorem 1.20 (First isomorphism theorem)**.** *Let $\varphi\colon R \to S$ be a surjective ring homomorphism. Then,*

$$R/\ker\varphi \cong \operatorname{im}\varphi.$$

Proof. Denote $I = \ker\varphi$, so the elements of R/I are the cosets $x + I$ for $x \in R$. This gives us the natural map

$$\phi\colon R/I \to S, \qquad x + I \mapsto \varphi(x).$$

It can be shown that this map is well defined: if $x + I = y + I$, then $x - y \in I$ so $\varphi(x-y) = 0$, or $\varphi(x) = \varphi(y)$. Now, $\phi((x+I)+(y+I)) = \varphi(x+y) = \varphi(x)+\varphi(y) = \phi(x+I)+\phi(y+I)$, and $\phi((x+I)(y+I)) = \varphi(xy) = \varphi(x)\varphi(y) = \phi(x+I)\phi(y+I)$. Additionally, if R and S are both unit rings, then $\phi(1_R + I) = \varphi(1_R) = 1_S$. Thus, ϕ is a homomorphism. It is obvious that ϕ is surjective; also observe that $\phi^{-1}(0) = 0 + I$, hence ϕ is also injective. This proves that ϕ is an isomorphism, as desired. □

Theorem 1.21. *Let $I, J \subset R$ be ideals. Then,*

$$(I+J)/J \cong I/(I \cap J).$$

Proof. The map $\phi\colon I \to (I+J)/J$, $x \mapsto x+J$ can be shown to be a surjective homomorphism. It's kernel consists of the elements in I that get mapped to $0+J$, so $\ker\phi = I \cap J$. Applying the first isomorphism theorem gives the desired result. □

Lemma 1.22. *Let $I \subset R$ be an ideal, and let $\varphi\colon R \to S$ be a surjective ring homomorphism, then $\varphi(I)$ is an ideal in S.*

Theorem 1.23 (Correspondence theorem)**.** *Let $I \subset R$ be an ideal. Then there exists a one-to-one correspondence between the ideals of R containing I with the ideals of R/I.*

Proof. Use the surjective ring homomorphism $\phi\colon R \to R/I$, $x \mapsto x+I$, which maps ideals in R to ideals in R/I. Furthermore, given ideals $J, J' \subset R$ such that $\varphi(J) = \varphi(J')$, note that $x \in J$ implies $\varphi(x) \in \varphi(J) = \varphi(J')$ so $x \in J'$; this shows that $J = J'$, hence our map is injective. Finally, given an ideal K in R/I, its pre-image under our map is the ideal $L = \{x \in R : x+I \in K\}$. □

Theorem 1.24 (Chinese remainder theorem)**.** *Let R be a commutative unit ring, and $I, J \subset R$ be ideals such that $I+J = R$. Then,*

$$R/IJ \cong R/I \times R/J.$$

Proof. Consider the map

$$\varphi\colon R \to R/I \times R/J, \qquad x \mapsto (x+I, x+J).$$

It is clear that this is a ring homomorphism. Furthermore, φ is surjective: to see this, pick $a \in I$, $b \in J$ such that $a+b=1$. Then

$$\varphi(ay+bx) = (a(y-x)+x+I, b(x-y)+y+J) = (x+I, y+J).$$

Now, note that $\varphi(x) = (I, J)$ forces $x \in I \cap J$; but the latter is just IJ by a previous lemma. Applying the first isomorphism theorem gives the desired result. □

1.7 Quotient fields

We recall the standard construction of $\mathbb{Q}$ from $\mathbb{Z}$, and generalize this to the construction of the field $Q(R)$ from an integral domain R. Consider the equivalence relation on the set $R \times R \setminus \{0\}$ defined by

$$(a,b) \sim (c,d) \iff ad = bc.$$

This partitions $R \times R \setminus \{0\}$ into equivalence classes; let $Q(R)$ be the collection of these equivalence classes. Now define addition and multiplication of elements from $Q(R)$ as

$$[a,b] + [c,d] = [ad+bc, bd], \qquad [a,b] \cdot [c,d] = [ac, bd].$$

It can be verified that this is well defined. Furthermore, we have an additive identity $[0,a]$, a multiplicative identity $[a,a]$, and every non-zero element $[a,b]$ has a multiplicative inverse $[b,a]$. The remaining properties can be checked to show that $Q(R)$ is a field. We can now embed R in $Q(R)$ via the map

$$\iota\colon R \to Q(R), \qquad x \mapsto [ax, a].$$

It can also be shown that $Q(R)$ is the smallest field containing R. Indeed if $j\colon R \to F$ is an embedding of R in the field F, we can embed $Q(R)$ in F using the map $[a,b] \mapsto j(a) \cdot j(b)^{-1}$.

Remark. We do not require R to have a multiplicative identity!

Definition 1.16. The field $Q(R)$ constructed as above is called the field of fractions, or quotient field of the integral domain R.

Lemma 1.25. *The field of fractions $Q(R)$ is the smallest field containing the integral domain R.*

Lemma 1.26. *Let R_1, R_2 be integral domains. If $R_1 \cong R_2$, then $Q(R_1) \cong Q(R_2)$.*

1.8 Prime and maximal ideals

Definition 1.17. An ideal $I \subseteq R$ is called a prime ideal if it is proper, and $xy \in I$ implies that at least one of $x, y \in I$ for all $x, y \in R$.

Lemma 1.27. *An ideal $I \subseteq P$ is prime if and only if $JK \subset I$ forces either $J \subset I$ or $K \subset I$ for all ideals $J, K \subseteq R$.*

Example. The prime ideals of $\mathbb{Z}$ are $\{0\}$ and $p\mathbb{Z}$

Example. A commutative ring is an integral domain if and only if $\{0\}$ is a prime ideal.

Theorem 1.28. *Let R be a commutative ring, and I be a proper ideal. Then, I is a prime ideal if and only if R/I is an integral domain.*

Example. The quotients $\mathbb{Z}/p\mathbb{Z}$ are integral domains precisely for primes p.

Definition 1.18. An ideal $I \subseteq R$ is called maximal if it is proper, and for any ideal $J \subseteq R$ with $I \subseteq J \subseteq R$, either $J = I$ or $J = R$.

Example. The maximal ideals of $\mathbb{Z}$ are $p\mathbb{Z}$.

Theorem 1.29. *Let R be a commutative unit ring, and I be a proper ideal. Then I is a maximal ideal if and only if R/I is a field.*

Example. Note that $4\mathbb{Z}$ is a maximal ideal in $2\mathbb{Z}$, but $2\mathbb{Z}/4\mathbb{Z}$ is not a field.

Lemma 1.30. *Let R be a commutative unit ring. Then every maximal ideal is prime.*

Example. Note that (X) is a prime ideal in $\mathbb{Z}[X]$, but not maximal.

Definition 1.19. A non-empty set S with a partial order $\leq$ is called a partial ordered set, when we have

1. $x \leq x$ for all $x \in S$.
2. $x \leq y$ and $y \leq x$ forces $x = y$.
3. $x \leq y$ and $y \leq z$ forces $x \leq z$.

Definition 1.20. A subset T of S is called a chain or totally ordered set if any two elements are comparable. In other words, given $x, y \in T$, at least one of $x \leq y$ or $y \leq x$.

Lemma 1.31 (Zorn's Lemma)**.** *If S is a partially ordered set such that every chain C has an upper bound in S, then for every element $x \in S$, there exists a maximal element $z \in S$ such that $x \leq z$.*

Theorem 1.32. *Let R be a commutative unit ring. Then R contains a maximal ideal.*

1.9 Divisibility

In this section, all rings are integral domains with a multiplicative identity.

Definition 1.21. Let $a, b \in R$, $a \neq 0$. We say that a divides b if there exists $c \in R$ such that $b = ac$. We denote this by $a \mid b$.

Example. In $\mathbb{Z}[i]$, $3 + i$ divides 10 because $10 = (3 + i)(3 - i)$.

Lemma 1.33. *If $a, b \in R$, $a \neq 0$, then $a \mid b$ if and only if $(a) \supseteq (b)$.*

Lemma 1.34. *Suppose that $a \mid b$ and $b \mid a$. Then, $b = ua$ for some unit $u \in R$.*

Definition 1.22. Two non-zero elements $a, b \in R$ are called associates of each other if $b = ua$ for some unit $u \in R$.

Remark. This defines an equivalence relation on $R - \{0\}$.

Definition 1.23. A non-zero non-unit element $a \in R$ is said to be irreducible if $a = bc$ forces either b, c to be a unit.

Remark. The only divisors of an irreducible element are its associates and units.

Definition 1.24. A non-zero non-unit element $p \in R$ is said to be prime if for $a, b \in R$, $p \mid ab$ forces either $p \mid a$, $p \mid b$.

Lemma 1.35. *All prime elements are irreducible.*

Example. Consider $x = 1 + \sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$; this is irreducible, but not prime.

Theorem 1.36. *Let $p \in R$ be non-zero. Then, p is a prime if and only if (p) is a prime ideal.*

Theorem 1.37. *Let $x \in R$ be non-zero. Then, x is irreducible if (x) is maximal.*

Example. Note that X is irreducible in $\mathbb{Z}[X]$, but (X) is not maximal.

Definition 1.25. Let $a, b \in R$ be non-zero. An element $d \in R$ is called a greatest common divisor (gcd) of a and b if

1. $d \mid a$ and $d \mid b$.
2. $d' \mid a$ and $d' \mid b$ forces $d' \mid d$.

Definition 1.26. Let $a, b \in R$ be non-zero. An element $l \in R$ is called a least common multiple (lcm) of a and b if

1. $a \mid l$ and $b \mid l$.
2. $a \mid l'$ and $b \mid l'$ forces $l \mid l'$.

Example. Consider the ring $\mathbb{Z}[\sqrt{-5}]$, with $a = 2(1+\sqrt{-5})$, $b = 6 = (1+\sqrt{-5})(1-\sqrt{-5})$. Then, a and b have no gcd or lcm.

Lemma 1.38. *If the gcd of a and b does exist, then it is unique upto associates. The same applies for the lcm.*

1.10 Factorisation domains

Definition 1.27. A unit integral domain R is called a factorisation domain if every non-zero, non-unit $x \in R$ can be expressed as a unit times a product of irreducible elements, i.e. $x = ux_1x_2 \dots x_n$ where u is a unit and each x_i is irreducible.

Example. The ring $\mathbb{Z}[\sqrt{-5}]$ is a factorisation domain.

Example. Consider the ring of entire complex functions, i.e.

$$R = \{\sum_{n=1}^{\infty} a_n z^n : a_n \in \mathbb{C}, \text{ the series converges for all } z \in \mathbb{C}\}.$$

Then, R is indeed a unit integral domain, and its units are those functions which vanish nowhere. Furthermore, its irreducible elements are the associates of linear polynomials $z - a$. Now if R were to be a factorisation domain, then every element would be an associate of a polynomial function, and thus have finitely many zeroes. However, the entire function sin has infinitely many zeroes.

Definition 1.28. A unit integral domain R is called a unique factorisation domain if R is a factorisation domain and the factorisation of every element non-zero $x \in R$ is unique upto associates.

Example. The ring of integers $\mathbb{Z}$ is a unique factorisation domain.

Example. The ring $\mathbb{Z}[\sqrt{-5}]$ is not a unique factorisation domain.

Theorem 1.39. *A unit integral domain R is a unique factorisation domain if and only if R is a factorisation domain in which every irreducible element is a prime.*

Proof. First suppose that R is a unique factorisation domain. Let x be an irreducible element in R, and let $x \mid ab$. We claim that $x \mid a$ or $x \mid b$. Now, $x \mid ab$ means that there exists $y \in R$, $xy = ab$. We can factor a and b, and conclude that

$$xy = ab = (ua_1 \dots a_l)(vb_1 \dots b_m),$$

where u, v are units and a_i, b_j are all irreducible. Since R is a unique factorisation domain and the irreducible element x appears on the left, it must be an associate of one of the a_i, b_j. If $x = wa_i$ for some unit w, then $x \mid a_i$ hence $x \mid a$. Otherwise, $x \mid b_j$ hence $x \mid b$.

Next suppose that R is a factorisation domain where every irreducible element is prime. Suppose that non-zero $x \in R$ factorises into irreducible elements as

$$x = ux_1 \dots x_l = vy_1 \dots y_m.$$

Note that all x_i, y_j are primes. Suppose that $l \leq m$. Now $x_1 \mid x$ implies that $x_1 \mid vy_1 \dots y_m$, hence $x_1 \mid y_k$ for some y_k. Without loss of generality, let $x_1 \mid y_1$; the irreducibility of y_1 means that $x_1 = u_1 y_1$ for some unit u_1. Thus,

$$uu_1 x_2 \dots x_l = vy_2 \dots y_m.$$

Continuing this process, we will reach $w = vy_{l+1} \dots y_m$ for some unit w, which is a contradiction if $l < m$. Thus, we are forced to have $l = m$, and all $x_i = u_i y_i$ for units u_i. □

Lemma 1.40. *Let R be a unique factorisation domain. Then, any two non-zero elements in R have a gcd and an lcm.*

1.11 Principal ideal domains

Definition 1.29. A unit integral domain R is called a principal ideal domain if every ideal of R is principal.

Example. The ring of integers $\mathbb{Z}$ is a principal ideal domain.

Theorem 1.41. *Let R be a principal ideal domain, and let $x \in R$ be non-zero. Then, x is irreducible if and only if (x) is maximal.*

Corollary 1.41.1. *Let R be a principal ideal domain. Then, every non-zero prime ideal is maximal.*

Example. Note that (0) is a prime ideal in $\mathbb{Z}$, but is not maximal.

Lemma 1.42. *Let R be a principal ideal domain. Then, every irreducible element is prime.*

Theorem 1.43. *Every principal ideal domain is a unique factorisation domain.*

Corollary 1.43.1. *Let R be a principal ideal domain. Then, any two non-zero elements in R have a gcd and an lcm.*

Example. The ring $\mathbb{Z}[X]$ is a unique factorisation domain, but not a principal ideal domain.

1.12 Euclidean domains

Definition 1.30. An integral domain R is called a Euclidean domain if there is a map $d\colon R - \{0\} \to \mathbb{Z}_{\geq 0}$ such that

1. $d(a) \leq d(ab)$ for all non-zero $a, b \in R$.
2. For all $a \in R$ and non-zero $b \in R$, there exist $q, r \in R$ such that $a = bq + r$ with either $r = 0$ or $d(r) < d(b)$.

The map d is called the algorithm map and the second property is called the division algorithm.

Example. The ring of integers $\mathbb{Z}$ is a Euclidean domain, with $d(n) = |n|$.

Example. The ring of Gaussian integers $\mathbb{Z}[i]$ is a Euclidean domain, with $d(a+ib) = a^2+b^2$.

Example. Every field is a Euclidean domain, with $d(x) = 1$.

Lemma 1.44. *Every ideal in a Euclidean domain is principal.*

Proof. Let R be a Euclidean domain, and let $I \subseteq R$ be an ideal. If $I = 0$, we trivially have $I = (0)$. Thus, let $I \neq 0$, and choose non-zero $a \in I$ such that $d(a)$ is minimal. We claim that $I = (a)$. Indeed, let $b \in I$, and exhibit $q, r \in R$ such that $b = aq + r$. This shows that $r = b - aq \in I$. Note that $d(r) < d(a)$ contradicts the minimality of $d(a)$, hence we must have $r = 0$, and $b = aq \in (a)$. Thus, $I = (a)$ as desired. □

Lemma 1.45. *Every Euclidean domain is a unit ring.*

Proof. The previous lemma shows that if R is a Euclidean domain, then $R = (a)$ for some $a \in R$. Since $a \in R$, we must have $a = a_0 a$ for some $a_0 \in R$. We claim that a_0 is the identity in R. Indeed, for $x \in R = (a)$, we must have $x = ra$ for some $r \in R$, hence $x = ra_0 a = a_0(ra) = a_0 x$. □

Theorem 1.46. *Every Euclidean domain is a principal ideal domain.*

Example. The ring $\mathbb{Z}[(1+\sqrt{19})/2]$ is a principal ideal domain, but not a Euclidean domain.

Corollary 1.46.1. *Every Euclidean domain is a unique factorisation domain.*

Corollary 1.46.2. *Let R be a Euclidean domain. Then, any two non-zero elements in R have a gcd and an lcm.*

Lemma 1.47. *Let R be a Euclidean domain, and let $a, b \in R$. If a is a proper divisor of b, then $d(a) < d(b)$.*

1.13 Polynomial rings

Theorem 1.48 (Eisenstein's criterion)**.** *Let R be a unique factorisation domain, and*

$$f(x) = \sum_{i=0}^{n} a_i x^i \in R[X].$$

Suppose that there is a prime $p \in R$ such that $a \mid a_i$ for $0 \leq i < n$, $p \nmid a_n$, and $p^2 \nmid a_0$. Then, $f(x)$ is irreducible.

Corollary 1.48.1. *Let p be a prime and let $n > 1$. Then, $x^n - p \in \mathbb{Z}[X]$ is irreducible.*

Lemma 1.49 (Gauss lemma)**.** *Let R be a unique factorisation domain, and let F be the field of fractions of R. Let $f(x) \in R[X]$ be irreducible in $\mathbb{R}[X]$. Then, $f(x)$ is irreducible in $F[X]$.*

Theorem 1.50 (Gauss theorem)**.** *Let R be a unique factorisation domain. Then, $R[X]$ is also a unique factorisation domain.*

2 Modules

2.1 Basic definitions

Definition 2.1. Let R be a ring. A left R-module is an abelian group $(M, +)$ together with a map $R \times M \to M$ given by $(a, x) \mapsto ax$ such that for all $a, b \in R$, $m, n \in M$, the following hold.

1. $a(m + n) = am + an$.
2. $(a + b)m = am + bm$.
3. $a(bm) = (ab)m$.
4. $1m = m$.

Remark. When R is commutative, we simply call this structure an R-module.

Example. Any abelian group is a $\mathbb{Z}$-module.

Example. Any vector space V over a field F is an F-module.

Example. Any ring R is an R-module over itself. Indeed, given an ideal $I \subseteq R$, we see that I is an R-module.

Example. Let M be an S-module. If $\varphi\colon R \to S$ is a ring homomorphism, we can treat M as an R-module, with scalar multiplication $(a, m) \mapsto \varphi(a)m$ for $a \in R$, $m \in M$.

In particular, S is an R-module, with $(r, s) \mapsto \varphi(r)s$ for $r \in R$, $s \in S$.

Using this idea, suppose that $I \subseteq R$ is an ideal, and $\pi\colon R \to R/I$ is the canonical homomorphism. Then R/I is an R-module, with $(r, s + I) \mapsto rs + I$ for $r, s \in R$.

2.2 Submodules

Definition 2.2. Let M be a module. A subset $N \subseteq M$ is called a submodule of M if N is a subgroup of $(M, +)$, and $rx \in N$ for all $r \in R$, $x \in N$.

Example. Any subspace of a vector space is a submodule.

Example. All polynomials of degree at most n is a submodule of the R module $R[x]$.

Example. The submodules of a commutative ring R are precisely the ideals of R.

Lemma 2.1. *Let $M_1, M_2 \subseteq M$ be submodules. Then, $M_1 \cap M_2$ is a submodule of M.*

Definition 2.3. Let $M_1, M_2 \subseteq M$ be submodules. The smallest submodule of M containing $M_1 \cup M_2$ is called the submodule generated by M_1, M_2.

Lemma 2.2. *The submodule generated by $M_1, M_2 \subseteq M$ is*

$$M_1 + M_2 = \{x + y : x \in M_1, y \in M_2\}.$$

2.3 Homomorphisms and isomorphisms

Definition 2.4. Let R be a ring, and let M, N be R-modules. A map $\varphi\colon M \to N$ is called a homomorphism if for all $r \in R$, $m, m' \in M$, the following hold.

1. $\varphi(m + m') = \varphi(m) + \varphi(m')$.
2. $\varphi(rm) = r\varphi(m)$.

The set of all R-module homomorphisms $\varphi\colon M \to N$ is denoted $\operatorname{Hom}_R(M, N)$.

Example. Any linear transformation between vector spaces over a field F is a homomorphism of F-modules.

Definition 2.5. A bijective homomorphism is called an isomorphism.

Definition 2.6. A homomorphism $\varphi\colon M \to M$ is called an endomorphism. The set of all R-module endomorphisms $\varphi\colon M \to N$ is denoted $\operatorname{End}_R(M, N) = \operatorname{Hom}_R(M, M)$.

Definition 2.7. A bijective endomorphism is called an automorphism.

Lemma 2.3. *Let $\varphi\colon M \to N$ be a homomorphism of R-modules. Then, the following hold.*

1. *The set $\ker \varphi$ is a submodule of M.*
2. *The set $\operatorname{im} \varphi$ is a submodule of N.*
3. *The map φ is a injective if and only if $\ker \varphi = \{0\}$.*

Theorem 2.4. *Let M be an R-module, and $N \subseteq M$ be a submodule. Then, the quotient group M/N is an R-module, with $r[m] \mapsto [rm]$ for $r \in R$, $m \in M$.*

Theorem 2.5. *If $\varphi \in \operatorname{Hom}_R(M, M')$, then $M/\ker \varphi \cong \operatorname{im} \varphi$.*

2.4 Cyclic modules

Definition 2.8. An R-module M is called cyclic if $M = Rx$ for some $x \in M$.

Definition 2.9. The annihilator of an R-module M is defined as

$$\operatorname{Ann}(M) = \{a \in R : aM = 0\}.$$

Definition 2.10. A cyclic module M is called free if $\operatorname{Ann}(M) = 0$.

Lemma 2.6. *If $M = Rx$ is free, then every element $z \in M$ can be uniquely expressed as $z = ax$ for $a \in R$.*

Definition 2.11. An R-module M is called finitely generated if $M = M_1 + M_2 + \cdots + M_n$ where each M_i is cyclic. If each $M_i = Rx_i$, then $\{x_1, \ldots, x_n\}$ is called a generating set of M.

Example. The module of polynomials over R of degree at most n is generated by $\{1, x, \ldots, x^n\}$.

2.5 Direct products and sums

Definition 2.12. Let I be an indexing set. A family $(x_i, i \in I)$ is a function whose value at i is x_i. Let M_i be R-modules. Their direct product $\prod_{i\in I} M_i$ is the set of all families $(x_i, i \in I)$ with $x_i \in M_i$. Addition and scalar multiplication are defined simply as

$$(x_i) + (y_i) = (x_i + y_i), \qquad r(x_i) = (rx_i),$$

Definition 2.13. The external direct sum of R-modules M_i, denoted $\bigoplus_{i\in I} M_i$, is the set of all families $(x_i, i \in I)$ where all but finitely many $x_i = 0$ for each index $i \in I$.

Definition 2.14. Let $M_1, M_2 \subseteq M$ be submodules. We say that M is the internal direct sum of M_1, M_2 if $M = M_1 + M_2$, $M_1 \cap M_2 = \{0\}$.

Lemma 2.7. *If M is the internal direct sum of M_1, M_2, then $M \cong M_1 \oplus M_2$.*

Definition 2.15. If M is an R-module and $M \cong M_1 \oplus M_2$, then this is called a direct decomposition of M. An indecomposable module M cannot be written as the direct sum of non-zero modules.

Definition 2.16. An R-module M is called free if on a finite basis, it can be expressed as a direct sum

$$M = M_1 \oplus \cdots \oplus M_n$$

where each M_i is a free cyclic R-module. If $M_i = Rx_i$, then the collection $\{x_1, \ldots, x_n\}$ is called a basis of the free module M.

Lemma 2.8. *If M is a free module with basis $\{x_1, \ldots, x_n\}$, then each $z \in M$ can be uniquely expressed as*

$$z = a_1 x_1 + \cdots + a_n x_n,$$

for $a_i \in R$.

Example. $\mathbb{R}^n$ is a free $\mathbb{R}$-module, with many choices of basis.

Example. $\mathbb{Z}/n\mathbb{Z}$ is not a free $\mathbb{Z}$-module, since $n \in \operatorname{Ann}(\mathbb{Z}/n\mathbb{Z})$.

Lemma 2.9. *Let R be a commutative unit ring. Then any two bases of a free R-module have the same number of elements.*

Definition 2.17. Let R be a commutative unit ring, and let F be a free R-module with a basis of n elements. Then, n is defined as the rank of F.

Lemma 2.10. *Let R be a principal ideal domain, and let M be a free R-module of rank n. If $N \subseteq M$ is a submodule, then N is free with rank $\leq n$.*

Corollary 2.10.1. *Any submodule of a finitely generated module over a principal ideal domain is finitely generated.*

Remark. Note that a finitely generated R-module M of rank n is isomorphic to R^n/K, where R^n is a free module of rank n.

Definition 2.18. An element $x \in M$ is a called a torsion element of the R-module M if $ax = 0$ for some non-zero $a \in R$.

Remark. The set of torsion elements in M, denoted $t(M)$, is a submodule of M when R is an integral domain.

Remark. If $t(M) = 0$, M is called a torsion-free module. If $t(M) = M$, M is called a torsion module.

Example. $\mathbb{Q} \oplus \mathbb{Z}/n\mathbb{Z}$ is neither a torsion module, nor torsion-free.

Theorem 2.11. *Let M be a finitely generated module over a principal ideal domain R. Then, M can be expressed as*

$$M = F \oplus t(M),$$

where F is a free R-module. Moreover, F is unique up to isomorphism.

Definition 2.19. Let $x \in M$ and let $\operatorname{Ann}(x) = (a) \subseteq R$. Then, a is called the period of x.

Remark. The period of an element $x \in M$ is unique up to units in R.

Definition 2.20. Let M be a finitely generated torsion module over a principal ideal domain. If $\operatorname{Ann}(M) = (c)$, then c is called the exponent of M.

Remark. The exponent of M is unique up to units in R.

Lemma 2.12. *Let M be a finitely generated torsion module over a principal ideal domain, with exponent $c = ab$ where $\gcd(a, b) = 1$. Then, $M = M_1 \oplus M_2$, where*

$$M_1 = \{x \in M : ax = 0\}, \qquad M_2 = \{x \in M : bx = 0\}.$$

Corollary 2.12.1. *Let M be a finitely generated torsion module over a principal ideal domain, with exponent $c = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ for primes $p_1, \dots, p_k$. Then, M can be expressed as*

$$M = M_1 \oplus \cdots \oplus M_k,$$

where each $M_i = \{x \in M : p_i^{\alpha_i} x = 0\}$.

Definition 2.21. A set of elements $\{y_1, \dots, y_n\} \subseteq M$ is called independent if $\sum_{i=1}^{m} a_i y_i = 0$ for $a_i \in R$ forces each $a_i y_i = 0$.

Remark. A linearly independent set is independent, but not all independent sets are linearly independent.

Lemma 2.13. *Let M be a finitely generated torsion module over a principal ideal domain, with exponent p^r where p is a prime, $r \geq 1$. Let $x_1 \in M$ with period p^r, let $\overline{M} = M/(x_1)$, and let $\overline{y}_1, \ldots, \overline{y}_m$ be independent elements of $\overline{M}$. Then, for each i, there exists a representative y_i of $\overline{y}_i$ such that the periods of y_i and $\overline{y}_i$ are the same. Furthermore, $x_1, y_1, \ldots, y_m$ are independent.*

Lemma 2.14. *Let M be a finitely generated torsion module over a principal ideal domain, with exponent p^r where p is prime, $r \geq 1$. Then, M can be expressed as a direct sum of cyclic modules*

$$M = R/(p^{r_1}) \oplus \cdots \oplus R/(p^{r_k}),$$

with $r_1 \geq \cdots \geq r_k \geq 1$.

Lemma 2.15. *Let R be a unique factorisation domain.*

1. *If $\gcd(a, b) = 1$, then $R/(ab) \cong R/(a) \oplus R/(b)$.*
2. *If $p \in R$ is prime and $b \in R$ is non-zero, then $R/(p) \cong bR/(bp)$.*

Theorem 2.16 (Structure theorem)**.** *Let M be a finitely generated module over a principal ideal domain R. Then, M can be expressed uniquely as*

$$M \cong F \oplus R/(q_1) \oplus \cdots \oplus R/(q_k),$$

where F is a free R-module and $q_1, \ldots, q_k$ are non-zero elements of R with $q_1 \mid \cdots \mid q_k$. Moreover, F is unique up to isomorphism, and $q_1, \ldots, q_k$ are unique up to units.

Corollary 2.16.1. *Let G be a finitely generated abelian group. Then, G can be expressed uniquely as*

$$G = F \oplus \mathbb{Z}/q_1 Z \oplus \cdots \oplus \mathbb{Z}/q_k\mathbb{Z},$$

where F is a free abelian group and $q_1, \ldots, q_k$ are non-zero, with $q_1 \mid \cdots \mid q_k$.

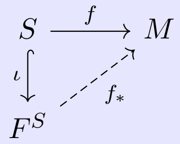
Remark. The rank of M is defined as the rank of F. The elements $q_1, \ldots, q_k$ are called the invariant factors of M.

Example. In order to enumerate all abelian groups G of order $60 = 2^2 \times 3 \times 5$ (up to isomorphism), observe that G has no free part. The only abelian groups of order 2^2 are $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the only abelian group of order 3 is $\mathbb{Z}/3\mathbb{Z}$, and the only abelian group of order 5 is $\mathbb{Z}/5\mathbb{Z}$. Thus, the invariant factors of G must be either 60, or $2, 30$. This gives us exactly two groups,

$$G \cong \mathbb{Z}/60\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z},$$
$$G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/30\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}.$$

2.6 Free modules

Theorem 2.17. *For any set S and a ring R, there exists a free module F^S and a map $\iota\colon S \hookrightarrow F^S$ such that $\iota(S)$ is a basis of F^S. This free module F^S is unique up to isomorphism. Furthermore, this has the following universal property: given any map $f\colon S \to M$ where M is an R-module, there exists a unique R-module homomorphism $f_*\colon F^S \to M$ such that $f = f_* \circ \iota$. In other words, each f induces a map f_* such that the following diagram commutes.*



Proof. Define

$$F^S = \{g\colon S \to R : g(a) = 0 \text{ for all but finitely many } a \in S\}.$$

This can be equipped with the R-module structure

$$(g+h)(a) = g(a) + h(a), \qquad (rg)(a) = rg(a)$$

for all $r \in R$, $g, h \in F^S$, $a \in S$. Now, define

$$\iota\colon S \to F^S, \qquad a \mapsto g_a,$$

where each $g_a(a) = 1$, $g_a(x) = 0$ for $x \neq a$. Then, it is clear that all such g_a generate F^S. Now, given $f\colon S \to M$, we may define

$$f_*\colon F^S \to M, \qquad g_a \mapsto f(a),$$

which is the required R-module homomorphism. □

2.7 Tensor products

Definition 2.22. Let R be a commutative unit ring, and let M, N, X be R-modules. A map $f\colon M \times N \to X$ is called R-bilinear if it is R-linear in each separate argument, i.e. for all $r \in R$, $x, y \in M$, $z, w \in N$, we have

1. $f(x+y, z) = f(x,z) + f(y,z)$.
2. $f(x, z+w) = f(x,z) + f(x,w)$.
3. $f(rx, z) = rf(x,z) = f(x, rz)$.

Definition 2.23. Let R be a commutative unit ring, and let M, N be R-modules. A tensor product of M and N is an R-module denoted $M \otimes_R N$, together with an R-bilinear map $\iota\colon M \times N \to M \otimes_R N$ such that for every R-bilinear map $f\colon M \times N \to X$, there is a unique R-linear map $f_*\colon M \otimes_R N \to X$ such that $f = f_* \circ \iota$. In other words, each f induces a map f_* such that the following diagram commutes.

$$\begin{array}{ccc} M \times N & \xrightarrow{f} & X \\ {\scriptstyle\iota}\downarrow & \nearrow_{f_*} & \\ M \otimes_R N & & \end{array}$$

We usually denote $\iota(x, y) \equiv x \otimes y$.

Remark. The R-module $M \otimes_R N$ is generated by the tensors $x \otimes y$.

Theorem 2.18. *The tensor product $M \otimes_R N$ of R-modules M, N exists, and is unique up to isomorphism.*

Example. We can show that $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, by picking an arbitrary element $x \times y$ and noting that

$$x \otimes y = n(x \otimes (y/n)) = (nx) \otimes (y/n) = 0 \otimes (y/n) = 0(0 \otimes (y/n)) = 0.$$

Example. We claim that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$. Consider the maps

$$\begin{aligned} \varphi\colon \mathbb{Q} &\to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}, & q &\mapsto q \otimes 1, \\ \psi\colon \mathbb{Q} \times \mathbb{Q} &\to \mathbb{Q}, & (p, q) &\mapsto pq. \end{aligned}$$

It is clear that φ is $\mathbb{Z}$-linear, ψ is $\mathbb{Z}$-bilinear hence ψ_* is $\mathbb{Z}$-linear. Furthermore, $\varphi \circ \psi_*$, $\psi_* \circ \varphi$ are identity maps on their respective domains.

Example. If $d = \gcd(m, n)$, then

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}.$$

To show this, we define the maps

$$\begin{aligned} \varphi\colon \mathbb{Z}/d\mathbb{Z} &\to \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}, & n &\mapsto n(1 \otimes 1), \\ \psi\colon \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} &\to \mathbb{Z}/d\mathbb{Z}, & (x, y) &\mapsto xy. \end{aligned}$$

It can be checked that these maps are indeed well-defined, φ is $\mathbb{Z}$-linear, ψ is $\mathbb{Z}$-bilinear hence ψ_* is $\mathbb{Z}$-linear. Furthermore, $\varphi \circ \psi_*$, $\psi_* \circ \varphi$ are identity maps on their respective domains.