# MA3202: Algebra II

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**Exercise 1** What is the field of fractions of a finite integral domain?

**Solution** Every finite integral domain is a field, hence its field of fractions is itself.

**Exercise 2** Check whether $4\mathbb{Z}$ and $6\mathbb{Z}$ are maximal ideals of $2\mathbb{Z}$.

**Solution** Suppose that $4\mathbb{Z} \subset I \subseteq 2\mathbb{Z}$, and $n \in I$, $n \notin 4\mathbb{Z}$. Then $n = 4k + 2$ for some $k \in Z$. Now, $4k \in 4\mathbb{Z} \subset I$, hence $(4k + 2) - 4k = 2 \in I$. This immediately gives all $2k \in I$, hence $I = 2\mathbb{Z}$. Thus, $4\mathbb{Z}$ is maximal in $2\mathbb{Z}$.

Suppose that $6\mathbb{Z} \subset I \subseteq 2\mathbb{Z}$, and $n \in I$, $n \notin 6\mathbb{Z}$. Then, $n = 6k \pm 2$ for some $k \in \mathbb{Z}$. A similar argument as before shows that $2 \in I$, $I = 2\mathbb{Z}$, thus showing that $6\mathbb{Z}$ is maximal in $2\mathbb{Z}$.

**Exercise 3** Let $R$ be a ring. What are the prime and maximal ideals of $R \times R$?

**Solution** Every prime ideal of $R \times R$ is of the form $P \times R$ or $R \times P$, for some prime ideal $P \subset R$. Similarly, every maximal ideal of $R \times R$ is of the form $M \times R$ or $R \times M$, for some maximal ideal $M \subset R$.

To see this, note that given an ideal $I \times J \subseteq R \times R$, the product of the quotient maps $R \to R/I$ and $R \to R/J$, given by the surjective homomorphism

$$\varphi\colon R \times R \to R/I \times R/J, \qquad (x, y) \mapsto (x + I, y + J),$$

has $I \times J$ as its kernel. Thus, $(R \times R)/(I \times J) \cong R/I \times R/J$. Now note that if $a \in R/I$, $b \in R/J$, then $(0, 0) = (a, 0) \cdot (0, b)$, hence we have zero divisors in the quotient unless one of $R/I$, $R/J$ is $\{0\}$, i.e. one of $I, J = R$. Thus, the quotient is now isomorphic to either $R/I$ or $R/J$, which must now be an integral domain/field forcing either $I, J$ to be prime/maximal.

**Exercise 4** Let $\varphi\colon R \to S$ be a surjective ring homomorphism. What can you say about the following?

(i) If $P \subset R$ is a prime ideal, then $\varphi(P)$ is a prime ideal of $S$.

(ii) If $M \subset R$ is a maximal ideal, then $\varphi(M)$ is a maximal ideal of $S$.

**Solution**

(i) False. Note that $(X) \subset \mathbb{R}[X]$ is a prime ideal, and the evaluation map $p(x) \mapsto p(1)$ is a surjective ring homomorphism $\mathbb{R}[X] \to \mathbb{R}$. However, $\varphi((X)) = \mathbb{R}$ is not a proper ideal of $\mathbb{R}$, hence not prime.

(ii) False, via the same counterexample. The ideal $(X) \subset \mathbb{R}[X]$ is indeed maximal.

**Exercise 5** Let $R$ be a commutative unit ring. Prove that $R[X]/(X)$ is isomorphic to $R$. Furthermore, the ideal $(X) = XR[X]$ is prime if and only if $R$ is an integral domain, and the ideal $(X) = XR[X]$ is maximal if and only if $R$ is a field.

**Solution** The map

$$\varphi\colon R[X] \to R, \qquad p(x) \mapsto p(0)$$

is a surjective homomorphism, with kernel $(X)$. This immediately gives $R[X]/(X) \cong R$. The next two criteria follow directly from the fact that $(X)$ is a proper ideal in a commutative unit ring, and $R[X]/(X) \cong R$.

**Exercise 6** Find all prime and maximal ideals of $\mathbb{Z}/16\mathbb{Z}$.

**Solution** Note that the ideals of $\mathbb{Z}/16\mathbb{Z}$ are $\mathbb{Z}/16\mathbb{Z}$, $2\mathbb{Z}/16\mathbb{Z}$, $4\mathbb{Z}/16\mathbb{Z}$, $8\mathbb{Z}/16\mathbb{Z}$, $\{0\}$. It is clear that $2\mathbb{Z}/16\mathbb{Z}$ is maximal, hence prime. $4\mathbb{Z}/16\mathbb{Z}$, $8\mathbb{Z}/16\mathbb{Z}$, $\{0\}$ are not prime since $4 = 2 \cdot 2$, $8 = 4 \cdot 2$, $0 = 4 \cdot 4$.

**Exercise 7** Let $R$ be a commutative unit ring such that $x^2 = x$ for all $x \in R$. Prove that every prime ideal of $R$ is a maximal ideal.

**Solution** Let $P \subset R$ be a prime ideal, and note that $R/P$ is an integral domain. Now, for any $x + P \in R/P$, we have

$$(x + P)^2 = x^2 + 2xP + P^2 = x + P,$$

hence every $y \in R/P$ satisfies $y^2 = y$, $y^2 - y = y(y-1) = 0$. Since $R/P$ is an integral domain, either $y = 0$ or $y = 1$. In other words, $R/P$ is the field of two elements, hence $P$ is a maximal ideal in $R$.

**Exercise 8** Let $\varphi\colon R \to S$ be a ring homomorphism. What can you say about the following?

(i) If $P \subset S$ is a prime ideal, then $\varphi^{-1}(P)$ is a prime ideal of $R$.

(ii) If $M \subset S$ is a maximal ideal, then $\varphi^{-1}(M)$ is a maximal ideal of $R$.

**Solution**

(i) False. Note that $2\mathbb{Z} \times \{0\} \subset 2\mathbb{Z} \times 2\mathbb{Z}$ is a prime ideal, and the map $n \mapsto (n, 0)$ is a ring homomorphism $2\mathbb{Z} \to 2\mathbb{Z} \times 2\mathbb{Z}$. However, $\varphi^{-1}(2\mathbb{Z} \times \{0\}) = 2\mathbb{Z}$ is not a proper ideal of $2\mathbb{Z}$, hence not prime.

(ii) False. Note that $\{0\} \subset \mathbb{Q}$ is a maximal ideal, and the inclusion map $\mathbb{Z} \to \mathbb{Q}$ is a ring homomorphism. However, the pre-image of $\{0\}$, namely $\{0\} \subset \mathbb{Z}$ is not maximal.

**Exercise 9** Let $Z[X]$ be the ring of polynomials with integer coefficients. Consider the ideal

$$J = \{f(x) \in \mathbb{Z}[X] : f(2) = 0\}.$$

(i) Is $J$ a prime ideal?

(ii) Is $J$ a maximal ideal?

**Solution** Note that $\mathbb{Z}[X]$ is a commutative unit ring, and the map $f(x) \to f(2)$ is a surjective ring homomorphism $\mathbb{Z}[X] \to \mathbb{Z}$, whose kernel is precisely $J$. Thus, $\mathbb{Z}[X]/J \cong \mathbb{Z}$, which is an integral domain but not a field. Thus, $J$ is a prime ideal but not maximal.

**Exercise 10** Show that an integral domain and its field of fractions have the same characteristic.

**Solution** Note that if $Z$ is an integral domain with field of fractions $F$, $Z$ is embedded in $F$. Thus if $Z$ has characteristic 0, so does $F$. Otherwise, let $n, m$ be the characteristics of $Z, F$ respectively. We immediately have $m \geq n$. Now, pick $[a, b] \in F$, hence $n[a, b] = [na, b] = [0, b] = 0 \in F$, which shows that $m \leq n$. Together, $m = n$ as desired.

**Exercise 11** Let $P, Q$ be two prime ideals in a ring $R$. Assume that $P \cap Q$ is a prime ideal of $R$. Prove that either $P \subset Q$ or $Q \subset P$.

**Solution** Suppose to the contrary that neither $P \subset Q$, nor $Q \subset P$. Then, we can pick $p \in P \setminus Q \subset P$, $q \in Q \setminus P \subset Q$, whence $pq \in P, Q$ from the properties of ideals. Thus, $pq \in P \cap Q$; since the latter is a prime ideal, we must have one of $p, q \in P \cap Q$, which is a contradiction.

**Exercise 12** Let $R = C([1, 3])$ and $I(2) = \{f \in R : f(2) = 0\}$.

(i) Prove that $I(2)$ is a maximal ideal.

(ii) Compute $R/I(2)$.

**Solution** Note that the evaluation map $f \mapsto f(2)$ is a surjective ring homomorphism $C([1,3]) \to \mathbb{R}$, whose kernel is precisely $I(2)$. Thus, $R/I(2) \cong \mathbb{R}$. Since $C([1,3])$ is a commutative unit ring and $\mathbb{R}$ is a field, $I(2)$ is a maximal ideal. It is clear that given an equivalence class in $R/I(2)$, all functions within that equivalence class have the same evaluation at 2. In other words,

$$R/I(2) = \{\alpha + I(2) : \alpha \in \mathbb{R}\}.$$