MA3202: Algebra II

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Exercise 1 Give an example of a non-commutative ring without identity.

Solution Consider $M_2(\mathbb{R}) \times 2\mathbb{Z}$. This is non-commutative, because

$$\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, 0\right) \cdot \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, 0\right) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, 0\right),$$

$$\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, 0\right) \cdot \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, 0\right) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, 0\right),$$

Also, this has no identity because for $(M, 2) \cdot (A, n) = (M, 2)$, we demand $2n = 2$ but there is no such identity element n in $2\mathbb{Z}$.

Exercise 2 Let R be a ring with 1 and $a \in R$. Suppose that there exists a positive integer n such that $a^n = 0$. Show that $1 + a$ and $1 - a$ are units.

Solution We use the factorizations

$$\begin{aligned} 1 = 1 - a^n &= (1 + a) \cdot (1 - a + a^2 - \cdots + (-1)^{n-1} a^{n-1}), \\ &= (1 - a) \cdot (1 + a + a^2 + \cdots + a^{n-1}). \end{aligned}$$

Exercise 3 Let R be a ring such that $r^2 = r$ for all $r \in R$. Show that R is commutative.

Solution Pick $x \in R$. Then,

$$2x = x + x = (x + x)^2 = 4x^2 = 4x, \qquad 2x = 0, \qquad x = -x.$$

Now, pick $x, y \in R$. Then,

$$x + y = (x + y)^2 = x^2 + xy + yx + y^2 = (xy + yx) + x + y,$$

whence

$$xy + yx = 0, \qquad xy = -yx = yx.$$

Exercise 4 Let R be a ring with 1, and $a, b \in R$. Prove that $1 - ab \in R^*$ if and only if $1 - ba \in R^*$.

Solution Suppose that $1 - ba$ is a unit, hence pick $x \in R$ such that $(1 - ba)x = 1$. This means that $x - bax = 1$, or $bax = x - 1$. Calculate

$$\begin{aligned} (1 - ab)(1 + axb) &= 1 + axb - ab - abaxb \\ &= 1 + axb - ab - a(x - 1)b \\ &= 1 + axb - ab - axb + ab \\ &= 1. \end{aligned}$$

Exercise 5 Does there exists a non-commutative ring with 77 elements?

Solution Let R be a ring with 77 elements. Then consider its additive group; Sylow's theorems tell us that there is exactly one 7-Sylow subgroup, and one 11-Sylow subgroup. Both of these are normal, hence $(R, +) \cong C_7 \times C_{11}$ is cyclic. In other words, we can pick a generator $a \in R$ such that every element is of the form na. It is now easy to check that $(ma) \cdot (na) = (mn)a = (na) \cdot (ma)$, hence R is commutative.

Exercise 6 Let R be a ring and let $R[x]$ be the polynomial ring over R. Show that $R[x]$ forms a ring over the usual addition and multiplication of polynomials.

Exercise 7 Does there exist an infinite ring with finite characteristic?

Solution Yes, the polynomial ring $\mathbb{Z}_2[x]$ has characteristic 2. Here, $\mathbb{Z}_n$ denotes $\mathbb{Z}/n\mathbb{Z}$.

Exercise 8 Does there exist a finite ring with characteristic zero?

Solution No; let R be a finite ring and pick non-zero $a \in R$. Now, examine the collection of elements $0, a, 2a, \ldots, na, \ldots$. Since R is finite, these cannot all be distinct. Thus, $na = ma$ for some $n < m$, hence $(m-n)a$.

Exercise 9 Determine the smallest subring of $\mathbb{Q}$ that contains 1/2.

Solution It is easy to check that this ring contains all elements of the form

$$\sum_{k=0^n} \frac{q_k}{2^k},$$

for $q_k \in \mathbb{Q}$, $n \geq 0$.

Exercise 10 Let R be an integral domain and $kx = 0$ for some non-zero $x \in R$ and some integer $k \neq 0$. Prove that R is of finite characteristic.

Solution Without loss of generality, let k be positive. Pick non-zero $y \in R$, and examine $k(xy)$. Factorizing this one way gives $(kx)y = 0$, and factoring this the other gives $x \cdot (ky)$. Since $x \neq 0$ and R is an integral domain, $x \cdot (ky) = 0$ forces $ky = 0$. Thus, every non-zero y has positive characteristic, hence R has finite characteristic as well.

Exercise 11 Give an example of a non-commutative simple ring.

Solution Consider the ring of quaternions, which is clearly non-commutative from $i^2 = j^2 = k^2 = ijk = -1$, hence $ij = k$, $ji = -k$. Let I be a non-trivial ideal, and let $a + bi + cj + dk \in I$ be non-zero. Then, note that

$$\begin{aligned}(a + bi + cj + dk) \cdot (x + yi + zj + wk) = ax &- by - cz - dw \\ &+ (ay + bx + cw - dz)i \\ &+ (az + cx + dy - bw)j \\ &+ (aw + dx + bz - cy)k.\end{aligned}$$

In particular,

$$(a + bi + cj + dk) \cdot (a - bi - cj - dk) = a^2 + b^2 + c^2 + d^2 \in I.$$

Since this is non-zero, we have $1 \in I$, forcing I to be the entire ring of quaternions. This shows that the ring of quaternions is simple.