The Trapezoidal Method

Numerically solving first order ordinary differential equations

Satvik Saha

November 23, 2021

Final project for MA3105: Numerical Analysis
Indian Institute of Science Education and Research, Kolkata

The initial value problem

The initial value problem

Given the derivative of a function y and an initial value $y(x_0)$, we wish to reconstruct the curve $y(x)$.

$$y'(x) = f(x, y(x)), \qquad y(x_0) = y_0.$$

For example,

$$y'(x) = y - x, \qquad y(0) = \frac{2}{3}.$$

The analytic solution

If

$$y'(x) = p(x)y(x) + q(x), \qquad y(x_0) = y_0,$$

then

$$y(x) = y_0 + g(x) \int_{x_0}^{x} \frac{q(x)}{g(x)}\, dx, \qquad g(x) = \exp \int_{a}^{x} p(x)\, dx.$$

In our example,

$$y(x) = \frac{2}{3} + e^x \int_0^x x e^{-x}\, dx = 1 + x - \frac{1}{3} e^x.$$

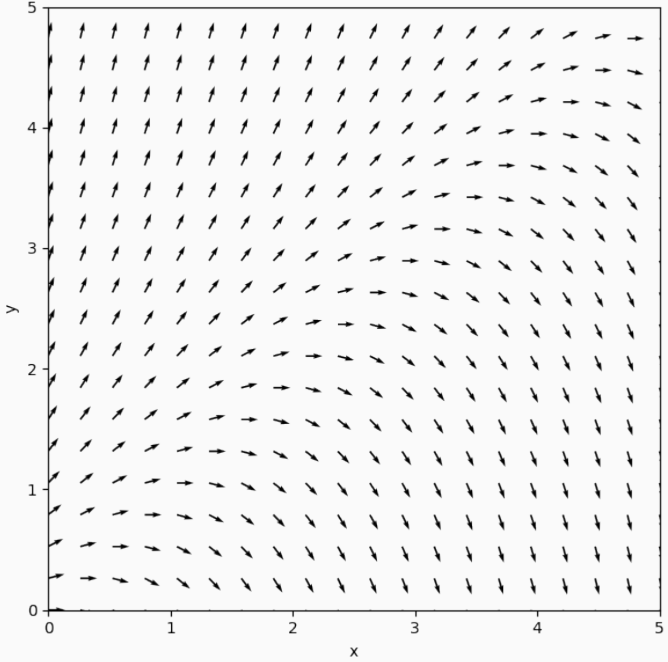
Examine the differential equation

$$y'(x) = f(x, y(x)).$$

This can be read as follows.

> *Given a point (x, y) on a solution curve $y(x)$, the tangent drawn to the curve at that point has slope $f(x, y)$.*

Thus, we can assign a vector with slope $f(x, y)$ to each point (x, y) on the plane; this gives us a *direction field*. Any solution $y(x)$ will fit neatly into this field.



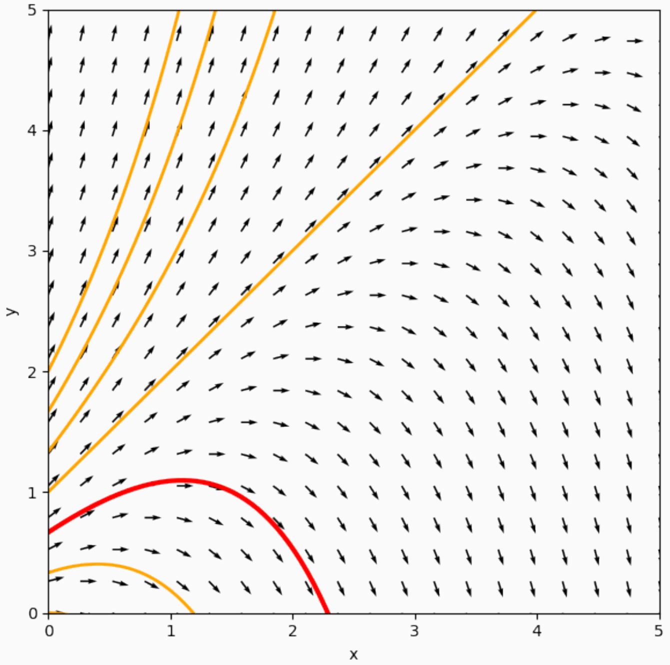
0
1
2
3
4
5
x
0
1
2
3
4
5
y

Joining the dots

1. Start at (x_0, y_0), and move along the tangent vector there by a small step to reach (x_1, y_1).
2. Join these points, and repeat.

The resulting curve closely approximates a solution to the initial value problem.

This ‘works’ because for a smooth curve $y(x)$, the tangent line closely approximates the curve.



0
1
2
3
4
5
x
y

Iterative methods

Euler's method

Recall that

$$y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}.$$

Thus, we can make the *forward* and *backward* approximations,

$$y'(x) \approx \frac{y(x+h) - y(x)}{h} \approx \frac{y(x) - y(x-h)}{h}.$$

Euler's method

This gives two ways of stepping along the x-axis, an explicit one and an implicit one.

$$y(x+h) \approx y(x) + h \cdot y'(x),$$
$$y(x+h) \approx y(x) + h \cdot y'(x+h).$$

More precisely, Taylor's theorem gives us

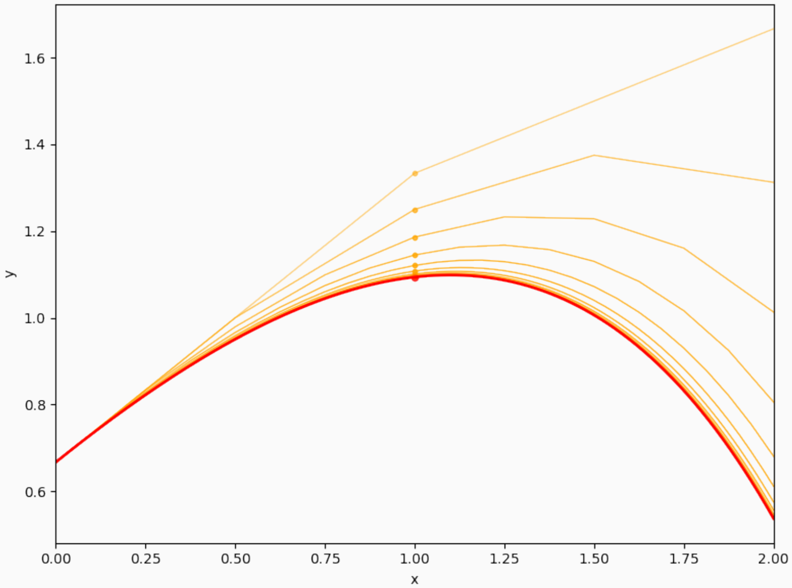
$$y(x+h) = y(x) + h \cdot y'(x) + \mathcal{O}(h^2).$$

Euler's method

Suppose that we wish to approximate the values of $y(x_i)$ for $x_i = x_0 + ih$, $0 \leq i \leq N$. Euler's method gives the scheme

$$y_{i+1} = y_i + h \cdot f(x_i, y_i).$$

We claim that the values $y_i \approx y(x_i)$.



1.6
1.4
1.2
1.0
0.8
0.6
y
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
x

Trapezoidal method

Recall our approximations

$$y(x+h) \approx y(x) + h \cdot y'(x),$$
$$y(x+h) \approx y(x) + h \cdot y'(x+h).$$

Taking their average,

$$y(x+h) \approx y(x) + \frac{1}{2}h \cdot \left[y'(x) + y'(x+h)\right].$$

Trapezoidal method

The following problems are essentially equivalent.

$$y'(x) = f(x, y(x)), \qquad y(0) = y_0,$$

$$\Updownarrow$$

$$y(x) = y_0 + \int_{x_0}^{x} f(x, y(x))\, dx.$$

The problem of integration can be split up into smaller pieces,

$$y(x_{i+1}) - y(x_i) = \int_{x_i}^{x_{i+1}} f(x, y(x))\, dx.$$

Trapezoidal method

$$
\begin{aligned}
y(x_{i+1}) - y(x_i) &= \int_{x_i}^{x_{i+1}} f(x, y(x))\, dx \\
&\approx \frac{1}{2} h \cdot \Big[f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1})) \Big].
\end{aligned}
$$

This uses the trapezoidal method of approximating integrals,

$$\int_a^b f(x)\, dx \approx \frac{b-a}{2} \cdot \Big[f(a) + f(b) \Big].$$

Trapezoidal method

The trapezoidal method gives the scheme

$$y_{i+1} = y_i + \frac{1}{2}h \cdot \Big[f(x_i, y_i) + f(x_{i+1}, y_{i+1})\Big].$$

This does *not* give y_{i+1} explicitly. Instead, we seek the root of

$$g(t) = -t + y_i + \frac{1}{2}h \cdot \Big[f(x_i, y_i) + f(x_{i+1}, t)\Big].$$

Solving the implicit equation

One way of solving for y_{i+1} is using Newton's method. Set up a good initial guess using Euler's method,

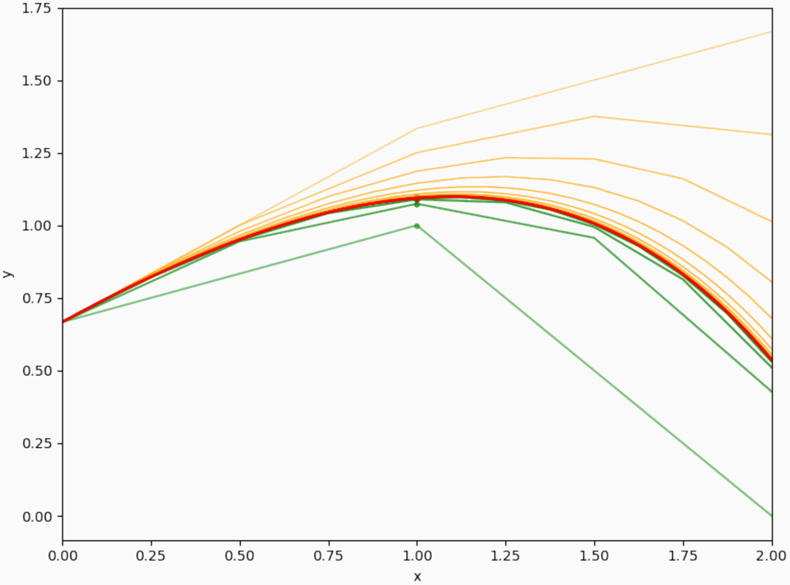
$$t_0 = y_i + h \cdot f(x_i, y_i),$$

and proceed with

$$t_{j+1} = t_j - \frac{g(t_j)}{g'(t_j)}.$$

We note that

$$g'(t) = -1 + \frac{1}{2} h \cdot \frac{\partial f}{\partial y}(x_{i+1}, t).$$



1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
y
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
x

An example by hand

Consider estimating $y(1)$ in one step (using $h = 1$) for the IVP

$$y'(x) = y - x, \qquad y(0) = \frac{2}{3}.$$

We solve the implicit equation for y_1.

$$y_1 = \frac{2}{3} + \frac{1}{2}\left[\frac{2}{3} + (y_1 - 1)\right], \qquad y_1 = 1.$$

Our analytic solution gives

$$y(1) = 2 - \frac{1}{3}e \approx 1.094,$$

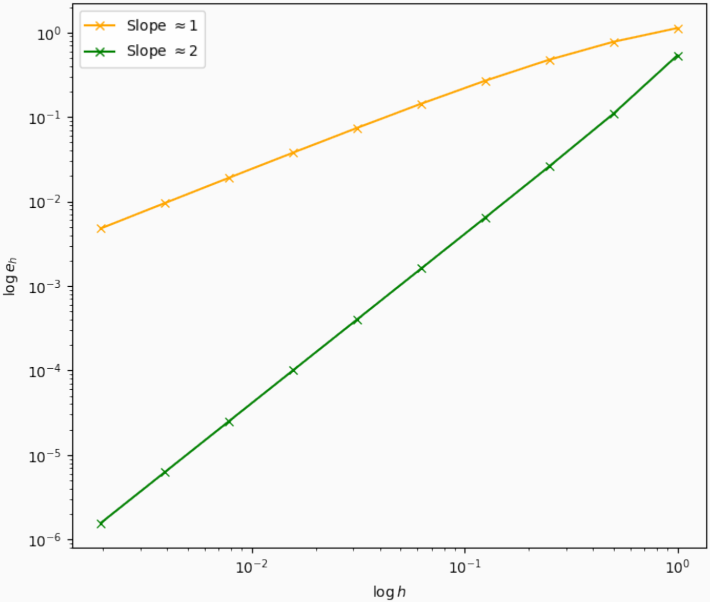
so we weren't far off!

An example by hand

Now try $h = 0.5$, so

$$y_1 = \frac{2}{3} + \frac{1}{4}\left[\frac{2}{3} + \left(y_1 - \frac{1}{2}\right)\right], \qquad y_1 = \frac{17}{18},$$
$$y_2 = \frac{17}{18} + \frac{1}{4}\left[\left(\frac{17}{18} - \frac{1}{2}\right) + (y_2 - 1)\right], \qquad y_2 = \frac{29}{27}.$$

This says $y_2 \approx 1.074$, off by 0.02 which is less than a quarter of the previous error.



Slope ≈ 1
Slope ≈ 2
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
$\log e_h$
10^{-2}
10^{-1}
10^0
$\log h$

Existence and uniqueness of solutions

Picard's theorem

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that the following properties hold.

1. f is continuous on $D = [x_0, x_N] \times [y_0 - C, y_0 + C]$.
2. $|f(x, y_0)| \leq K$ for $x \in [x_0, x_N]$.
3. f is Lipschitz in the second variable, i.e. there exists $L > 0$ such that for all $x \in [x_0, x_N]$, $u, v \in [y_0 - C, y_0 + C]$, we have
$$|f(x, u) - f(x, v)| \leq L|u - v|.$$
4. $$C \geq \frac{K}{L}\left(e^{L(x_N - x_0)} - 1\right).$$

Picard's theorem

Then, there exists a unique function $y \in C^1[x_0, x_N]$ solving the IVP

1. $y(x_0) = y_0$.
2. $y'(x) = f(x, y)$, on $[x_0, x_N]$
3. $|y(x) - y_0| \leq C$ on $[x_0, x_N]$

Convergence of iterative methods

General one-step iterative methods

Consider the scheme

$$y_{i+1} = y_i + h \cdot \Phi(x_i, y_i; h).$$

Here, Φ is continuous in all its variables.

For example, in Euler's method,

$$\Phi(x, y; h) = f(x, y).$$

In the trapezoidal method,

$$\Phi(x, y; h) = \frac{1}{2}\Big[f(x, y) + f(x + h, y + h \cdot \Phi(x, y; h))\Big].$$

Global and truncation errors

Define the global errors

$$e_i = y(x_i) - y_i.$$

Define the truncation errors

$$T_i = \frac{y(x_{i+1}) - y(x_i)}{h} - \Phi(x_i, y(x_i); h).$$

Global and truncation errors

Let Φ be Lipschitz in its second variable, i.e. there exists $L_\Phi > 0$ such that for all $0 \leq h \leq h_0$,

$$|\Phi(x, u; h) - \Phi(x, v; h)| \leq L_\Phi |u - v|.$$

Then, assuming that all $|y_i - y_0| \leq C$,

$$|e_n| \leq \frac{T}{L_\Phi} \left(e^{L_\Phi (x_n - x_0)} - 1 \right),$$

where $T = \max_{0 \leq i < n} |T_i|$.

Global and truncation errors

The truncation formula can be rearranged as

$$y(x_{i+1}) = y(x_i) + h \cdot \Phi(x_i, y(x_i); h) + hT_i.$$

Subtracting the iteration scheme, we have

$$e_{i+1} = e_i + h \cdot \Big[\Phi(x_i, y(x_i); h) - \Phi(x_i, y_i; h)\Big] + hT_i.$$

Use the Lipschitz condition to estimate the bracketed term.

$$\begin{aligned} |e_{i+1}| &\le |e_i| + hL_\Phi|e_i| + h|T_i| \\ &\le (1 + hL_\Phi)|e_i| + hT. \end{aligned}$$

Global and truncation errors

Denote $r = 1 + hL_{\Phi}$.

$$\begin{aligned}
|e_0| &= 0, \\
|e_1| &\leq hT \\
|e_2| &\leq rhT + hT = (r+1)hT \\
|e_3| &\leq r(rhT + hT) + hT = (r^2 + r + 1)hT \\
&\vdots \\
|e_n| &\leq (r^{n-1} + r^{n-1} + \cdots + r + 1)hT = hT \cdot \frac{r^n - 1}{r - 1}.
\end{aligned}$$

Global and truncation errors

Use $r = 1 + hL_\Phi \leq e^{hL_\Phi}$, $x_n = x_0 + nh$.

$$|e_n| \leq hT \cdot \frac{e^{nhL_\Phi} - 1}{hL_\Phi} = \frac{T}{L_\Phi}\left(e^{L_\Phi(x_n - x_0)} - 1\right).$$

Consistency

We demand that the truncation errors vanish as $h \to 0$. In other words, our numerical method is said to be consistent with the given ODE if for any $\epsilon > 0$, there exists $h_\epsilon > 0$ such that for all $0 \le h \le h_\epsilon$, we have $|T_i| \le \epsilon$ for any choice of $0 \le i < N$, any solution curve $y(x)$.

$$T_i = \frac{y(x_{i+1}) - y(x_i)}{h} - \Phi(x_i, y(x_i); h).$$

Let $h \to 0$, $N \to \infty$, such that $x_i \to x$. Using the continuity of y, y', Φ, we have

$$0 = y'(x) - \Phi(x, y(x); 0), \qquad \Phi(x, y; 0) = f(x, y).$$

Convergence

Let our IVP satisfy the conditions of Picard's theorem, let the one-step method generate approximations in the region D for all $h \leq h_0$. Recall that Φ is continuous in all its variables, and Lipschitz in the second variable. Also suppose that the consistency condition is satisfied. Then, the successive approximation sequences (y_i), generated using finer and finer meshes (decreasing h) converge to the solution of the IVP.

As $h \to 0$, pick points $x_n \to x \in [x_0, x_N]$ as $n \to \infty$. Then, the corresponding $y_n \to y(x)$.

Convergence

Choose $h \leq h_0$, such that there are N mesh points. Then,

$$|y(x_n) - y_n| \leq \frac{T}{L_\Phi}\left(e^{L_\Phi(x_N - x_0)} - 1\right).$$

Use consistency to write

$$\begin{aligned} T_n &= \frac{y(x_{n+1}) - y(x_n)}{h} - \Phi(x_n, y(x_n); h) \\ &= \frac{y(x_{n+1}) - y(x_n)}{h} - f(x_n, y(x_n)) + \\ &\qquad \Phi(x_n, y(x_n); 0) - \Phi(x_n, y(x_n); h) \\ &= \left[\frac{y(x_{n+1}) - y(x_n)}{h} - y'(x_n)\right] + \\ &\qquad \left[\Phi(x_n, y(x_n); 0) - \Phi(x_n, y(x_n); h)\right]. \end{aligned}$$

Use the Mean Value theorem to choose $\xi_n \in [x_n, x_{n+1}]$ such that $y(x_{n+1}) - y(x_n) = hy'(\xi_n)$.

Note that a continuous function on a compact set is also uniformly continuous. Thus, we can choose h_1 such that for all $h \leq h_1$,

$$|y'(\xi_n) - y'(x_n)| \leq \frac{1}{2}\epsilon,$$

and choose h_2 such that for all $h \leq h_2$,

$$|\Phi(x_n, y(x_n); 0) - \Phi(x_n, y(x_n); h)| \leq \frac{1}{2}\epsilon.$$

Putting $h_\epsilon = \min(h_1, h_2)$, we see that for all $h \leq h_\epsilon$, $|T_n| \leq \epsilon$.

Thus,

$$\begin{aligned} |y(x) - y_n| &\leq |y(x) - y(x_n)| + |y(x_n) - y_n| \\ &\leq |y(x) - y(x_n)| + \frac{\epsilon}{L_\Phi}\left(e^{L_\Phi(x_N - x_0)} - 1\right). \end{aligned}$$

As $n \to \infty$, $x_n \to x$, the continuity of y gives $y(x_n) \to y(x)$, making the first term vanish. By making ϵ arbitrarily small, the second term also vanishes. This gives $y_n \to y(x)$, as desired.

Application to Euler's method

We have

$$\Phi(x, y; h) = f(x, y).$$

thus

$$T_i = \frac{y(x_{i+1}) - y(x_i)}{h} - y'(x_i).$$

Assume y is twice continuously differentiable; use Taylor's theorem to conclude that

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{1}{2}h^2 y''(\xi_i)$$

for $\xi \in [x_i, x_{i+1}]$, hence

$$T_i = \frac{1}{2}hy''(\xi_i), \qquad T \leq \frac{1}{2}h \sup |y''(\xi)|.$$

Thus, the global error obeys

$$|e_n| \propto T = \mathcal{O}(h).$$

Application to the Trapezoidal method

We have

$$\Phi(x, y; h) = \frac{1}{2}\Big[f(x, y) + f(x + h, y + h \cdot \Phi(x, y; h))\Big].$$

To see that Φ satisfies the Lipschitz condition, we compute

$$\begin{aligned}
&|\Phi(x, u; h) - \Phi(x, v; h)| \\
&\qquad \leq \frac{1}{2}|f(x, u) - f(x, v)| + \\
&\qquad\qquad \frac{1}{2}|f(x, u + h\Phi(x, u; h)) - f(x, v + h\Phi(x, v; h))| \\
&\qquad \leq \frac{1}{2}L|u - v| + \frac{1}{2}L|u - v| + \frac{1}{2}Lh|\Phi(x, u; h) - \Phi(x, v; h)|.
\end{aligned}$$

Application to the Trapezoidal method

Rearranging, we see that

$$|\Phi(x, u; h) - \Phi(x, v; h)| \leq \frac{L}{1 - Lh/2}|u - v|.$$

For sufficiently small h, we have $Lh/2 < 1$ so choose

$$L_\Phi \leq \frac{L}{1 - Lh/2}.$$

Application to the Trapezoidal method

Now, compute the truncation error

$$
\begin{aligned}
T_i = \frac{y(x_{i+1}) - y(x_i)}{h} - \frac{1}{2}\Big[f(x_i, y(x_i)) + \\
f(x_i + h, y(x_i) + h\Phi(x_i, y(x_i); h))\Big] \\
= \frac{y(x_{i+1}) - y(x_i)}{h} - \frac{1}{2}\Big[f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1}))\Big] + \\
\frac{1}{2}\Big[f(x_{i+1}, y(x_{i+1})) - f(x_{i+1}, y(x_i) + h\Phi(x_i, y(x_i); h))\Big]
\end{aligned}
$$

Application to the Trapezoidal method

Note that if y is thrice continuously differentiable,

$$
\begin{aligned}
&\frac{y(x_{i+1}) - y(x_i)}{h} - \frac{1}{2}\Big[f(x_i, y(x_i)) + f(x_{i+1}, y(x_{i+1}))\Big] \\
&= \frac{y(x_{i+1}) - y(x_i)}{h} - \frac{1}{2}\Big[y'(x_i) + y'(x_{i+1})\Big] \\
&= y'(x_i) + \frac{1}{2}hy''(x_i) + \frac{1}{6}h^2y'''(\xi_i) - \\
&\qquad \frac{1}{2}\Big[y'(x_i) + y'(x_i) + hy''(x_i) + \frac{1}{2}h^2y'''(\zeta_i)\Big] \\
&= \frac{1}{12}h^2\Big[2y'''(\xi_i) - 3y'''(\zeta_i)\Big].
\end{aligned}
$$

Application to the Trapezoidal method

For the final term, use the Lipschitz condition to estimate

$$
\begin{aligned}
&\frac{1}{2}|f(x_{i+1}, y(x_{i+1})) - f(x_{i+1}, y(x_i) + h\Phi(x_i, y(x_i); h))| \\
\leq\ &\frac{1}{2}L|y(x_{i+1}) - y(x_i) - h\Phi(x_i, y(x_i); h)| \\
=\ &\frac{1}{2}Lh|T_i|.
\end{aligned}
$$

Application to the Trapezoidal method

Further imposing $Lh/2 < 1/2$, we can put these together to get

$$|T_i| \leq \frac{1}{12}h^2|2y'''(\xi_i) - 3y'''(\zeta_i)| + \frac{1}{2}Lh|T_i|,$$
$$T \leq \frac{5}{6}h^2 \sup|y'''(\zeta)|.$$

Thus, the global error obeys

$$|e_n| \propto T = \mathcal{O}(h^2).$$