MA3101

Analysis III

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Contents

1 Euclidean spaces

1.1 $\mathbb{R}^n$ as a vector space

We are familiar with the vector space $\mathbb{R}^n$, with the standard inner product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 + \cdots + x_n y_n.$$

The standard norm is defined as

$$\|\boldsymbol{x} - \boldsymbol{y}\|^2 = \langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{x} - \boldsymbol{y} \rangle = \sum_{k=1}^{n} (x_i - y_i)^2.$$

Exercise 1.1. What are all possible inner products on $\mathbb{R}^n$?

Solution. Note that an inner product is a bilinear, symmetric map such that $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \geq 0$, and $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ if and only if $\boldsymbol{x} = \boldsymbol{0}$. Thus, an product map on $\mathbb{R}^n$ is completely and uniquely determined by the values $\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = a_{ij}$. Let A be the $n \times n$ matrix with entries a_{ij}. Note that A is a real symmetric matrix with positive entries. Now,

$$\langle \boldsymbol{x}, \boldsymbol{e}_j \rangle = x_1 a_{1j} + \cdots + x_n a_{nj} = \boldsymbol{x}^\top \boldsymbol{a}_j,$$

where $\boldsymbol{a}_j$ is the j^{th} column of A. Thus,

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^\top \boldsymbol{a}_1 y_1 + \cdots + \boldsymbol{x}^\top \boldsymbol{a}_n y_n = \boldsymbol{x}^\top A \boldsymbol{y}.$$

Furthermore, any choice of real symmetric A with positive entries produces an inner product.

Theorem 1.1 (Cauchy-Schwarz)**.** *Given two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$, we have*

$$|\langle \boldsymbol{v}, \boldsymbol{w} \rangle| \leq \|\boldsymbol{v}\| \|\boldsymbol{w}\|.$$

Proof. This is trivial when $\boldsymbol{w} = \boldsymbol{0}$. When $\boldsymbol{w} \neq \boldsymbol{0}$, set $\lambda = \langle \boldsymbol{v}, \boldsymbol{w} \rangle / \|\boldsymbol{w}\|^2$. Thus,

$$0 \leq \|\boldsymbol{v} - \lambda \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 - 2\lambda \langle \boldsymbol{v}, \boldsymbol{w} \rangle + \lambda^2 \|\boldsymbol{w}\|^2.$$

Simplifying,

$$0 \leq \|\boldsymbol{v}\|^2 - \frac{|\langle \boldsymbol{v}, \boldsymbol{w} \rangle|^2}{\|\boldsymbol{w}\|^2}.$$

This gives the desired result. Clearly, equality holds if and only if $\boldsymbol{v} = \lambda \boldsymbol{w}$. □

Theorem 1.2 (Triangle inequality)**.** *Given two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$, we have*

$$\|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|.$$

Proof. Write

$$\|\boldsymbol{v} + \boldsymbol{w}\|^2 = \|\boldsymbol{v}\|^2 + 2\langle \boldsymbol{v}, \boldsymbol{w} \rangle + \|\boldsymbol{w}\|^2 \leq \|\boldsymbol{v}\|^2 + 2|\langle \boldsymbol{v}, \boldsymbol{w} \rangle| + \|\boldsymbol{w}\|^2.$$

Applying Cauchy-Schwarz gives

$$\|\boldsymbol{v} + \boldsymbol{w}\|^2 \leq (\|\boldsymbol{v}\| + \|\boldsymbol{w}\|)^2.$$

Equality holds if and only if $\boldsymbol{v} = \lambda \boldsymbol{w}$ for $\lambda \geq 0$. □

1.2 $\mathbb{R}^n$ as a metric space

Our previous observations allow us to define the standard metric on $\mathbb{R}^n$, seen as a point set.

$$d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|.$$

Definition 1.1. For any $\delta > 0$, the set

$$B_\delta(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{R}^n : d(\boldsymbol{x}, \boldsymbol{y}) < \delta\}$$

is called the open ball centred at $\boldsymbol{x} \in \mathbb{R}^n$ with radius δ. This is also called the δ neighbourhood of $\boldsymbol{x}$.

Definition 1.2. A set U is open in $\mathbb{R}^n$ if for every $\boldsymbol{x} \in U$, there exists an open ball $B_\delta(\boldsymbol{x}) \subset U$.

Remark. Every open ball in $\mathbb{R}^n$ is open.

Remark. Both $\emptyset$ and $\mathbb{R}^n$ are open.

Definition 1.3. A set F is closed in $\mathbb{R}^n$ if its complement $\mathbb{R}^n \setminus F$ is open in $\mathbb{R}^n$.

Remark. Both $\emptyset$ and $\mathbb{R}^n$ are closed.

Remark. Finite sets in $\mathbb{R}^n$ are closed.

Theorem 1.3. *Unions and finite intersections of open sets are open.*

Corollary 1.3.1. *Intersections and finite unions of closed sets are closed.*

Definition 1.4. An interior point x of a set $S \subseteq \mathbb{R}^n$ is such that there is a neighbourhood of x contained within S.

Example. Every point in an open set is an interior point by definition. The interior of a set is the largest open set contained within it.

Definition 1.5. An exterior point x of a set $S \subseteq \mathbb{R}^n$ is an interior point of the complement $\mathbb{R}^n \setminus S$.

Definition 1.6. A boundary point of a set is neither an interior point, nor an exterior point.

Example. The boundary of the unit open ball $B_1(0) \subset \mathbb{R}^n$ is the sphere S^{n-1}.

Definition 1.7. A limit point x of a set $S \subseteq \mathbb{R}^n$ is such that every neighbourhood of x contains a point from S other than itself.

Definition 1.8. The closure of a set $S \subseteq \mathbb{R}^n$ is the union of S and its limit points.

Remark. The closure of a set is the smallest closed set containing it.

Lemma 1.4. *Every open set in $\mathbb{R}^n$ is a union of open balls.*

Proof. Let $U \subseteq \mathbb{R}^n$ be open. Thus, for every $\boldsymbol{x} \in \mathbb{R}^n$, we can choose $\delta_x > 0$ such that $B_{\delta_x}(\boldsymbol{x}) \subset U$. The union of all such open balls is precisely the set U. $\square$

1.3 $\mathbb{R}^n$ as a topological space

Definition 1.9. A topology on a set X is a collection τ of subsets of X such that

1. $\emptyset \in \tau$
2. $X \in \tau$
3. Arbitrary union of sets from τ belong to τ.
4. Finite intersections of sets from τ belong to τ.

Sets from τ are called open sets.

Example. The Euclidean metric induces the standard topology on $\mathbb{R}^n$.

Example. The discrete topology on a set X is one where every singleton set is open. This is the topology induced by the discrete metric,

$$d_{\text{discrete}} \colon X \times X \to \mathbb{R}, \qquad (x, y) \mapsto \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y. \end{cases}$$

Example. Let X be an infinite set. The collection of sets consisting of $\emptyset$ along with all sets A such that $X \setminus A$ is finite is a topology on X. This is called the Zariski topology.

Example. Consider the set of real numbers, and let τ be the collection $\emptyset$, $\mathbb{R}$, and all intervals $(-x, +x)$ for $x > 0$. This constitutes a topology on $\mathbb{R}$, very different from the usual one.

This topology cannot be induced by a metric; it is not metrizable.

Consider the constant sequence of zeros. In this topology $(\mathbb{R}, \tau)$, this sequence converges to *every* point in $\mathbb{R}$. Given any $\ell \in \mathbb{R}$, the open neighbourhoods of ℓ are precisely the sets $\mathbb{R}$ and the open intervals $(-x, +x)$ for $x > |\ell|$. The tail of the constant sequence of zeros is contained within every such neighbourhood of ℓ, hence $0 \to \ell$. Indeed, the element zero belongs to every open set apart from $\emptyset$ in this topology.

Definition 1.10. A topological space is called Hausdorff if for every distinct $x, y \in X$, there exist disjoint neighbourhoods of x and y.

Example. Every metric space is Hausdorff. Given distinct x, y in a metric space (X, d), set $\delta = d(x, y)/3$ and consider the open balls $B_\delta(x)$ and $B_\delta(y)$.

Lemma 1.5. *Every convergent sequence in a Hausdorff space has exactly one limit.*

Proof. Consider a sequence $\{x_n\}_{n\in\mathbb{N}}$, and suppose that it converges to distinct x_1 and x_2. Construct disjoint neighbourhoods U_1 and U_2 around x_1 and x_2. Now, convergence implies that both U_1 and U_2 contain the tail of $\{x_n\}$, which is impossible since they are disjoint and hence contain no elements in common. □

Definition 1.11. Given a topological space (X, τ) and a subset $Y \subseteq X$, the collection of sets $U \cap Y$ where $U \in \tau$ is a topology τ_Y on Y. We call this collection the subspace topology on Y, induced by the topology on X.

1.4 Compact sets in $\mathbb{R}^n$

Definition 1.12. A set $K \subset X$ in a topological space is compact if every open cover of K has a finite sub-cover. That is, for every collection if $\{U_\alpha\}_{\alpha\in A}$ of open sets such that K is contained in their union, there exists a finite sub-collection $U_{\alpha_1}, \dots, U_{\alpha_k}$ such that K is also contained in their union.

Example. All finite sets are compact.

Example. Given a convergent sequence of real numbers $x_n \to x$, the collection $\{x_n\}_{n\in\mathbb{N}} \cup \{x\}$ is compact.

Example. In $\mathbb{R}^n$, compact sets are precisely those sets which are closed and bounded. This is the Heine-Borel Theorem.

Theorem 1.6. *The closed intervals $[a, b] \subset \mathbb{R}$ are compact.*

Remark. This can be extended to show that any k-cell $[a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ is compact.

Proof. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of $[a, b]$, and suppose that $I_1 = [a, b]$ has no finite sub-cover. Then, at least one of the intervals $[a, (a + b)/2]$ and $[(a + b)/2, b]$ must not have a finite sub-cover; pick one and call it I_2. Similarly, one of the halves of I_2 must not have a finite sub-cover; call it I_3. In this process, we generate a sequence of closed intervals $I_1 \supset I_2 \supset \dots$, none of which have a finite sub-cover. The length of each interval is given by

$$|I_n| = 2^{-n+1}\|b-a\| \to 0.$$

Now, pick a sequence of points $\{x_n\}$ where each $x_n \in I_n$. Then, $\{x_n\}$ is a Cauchy sequence. To see this, given any $\epsilon > 0$, we can find sufficiently large n_0 such that $2^{-n_0+1}\|b-a\| < \epsilon$. Thus, $x_n \in I_n \subset I_{n_0}$ for all $n \geq n_0$, which means that for any $m, n \geq n_0$, we have $x_m, x_n \in I_{n_0}$ forcing[1]

$$\|x_m - x_n\| \leq |I_{n_0}| = 2^{-n_0+1}\|b-a\| < \epsilon.$$

From the completeness of $\mathbb{R}$, this sequence must converge in $\mathbb{R}$, specifically in $[a,b]$. Thus, $x_n \to x$ for some $x \in [a,b]$. It can also be seen that the limit $x \in I_n$ for all $n \in \mathbb{N}$; if not, say $x \notin I_{n_0}$, then $x \in [a,b] \setminus I_{n_0}$ which is open, hence there is an open interval such that $(x-\delta, x+\delta) \cap I_{n_0} = \emptyset$. However, I_{n_0} contains all $x_{n \geq n_0}$, thus this δ-neighbourhood of x would miss out a tail of $\{x_n\}$.

Now, pick the open set $U \in \{U_\alpha\}$ which covers the point x. Thus, $x \in U$ so U contains some non-empty open interval $(x-\delta, x+\delta)$ around x. Choose n_0 such that $2^{-n_0+1}\|b-a\| < \delta$; this immediately gives $I_{n_0} \subseteq (x-\delta, x+\delta) \subset U$. This contradicts that fact that I_{n_0} has no finite sub-cover from $\{U_\alpha\}$, completing the proof. □

Remark. The fact that Cauchy sequences in $\mathbb{R}^n$ converge isn't immediately obvious; it is a consequence of the completeness of $\mathbb{R}^n$. Start by noting that $\mathbb{R}$ has the Least Upper Bound property, from which the Monotone Convergence Theorem follows; every monotonic, bounded sequence of reals converges. It can also be shown that any sequence of reals with contain a monotone subsequence, from which it follows that every bounded sequence contains a convergent subsequence (Bolzano-Weierstrass). Finally, it can be shown that if a subsequence of a Cauchy sequence converges, then the entire sequence also converges to the same limit, giving us the desired result for $\mathbb{R}$. For sequence in $\mathbb{R}^n$, we may apply this coordinate-wise to obtain the result.

Lemma 1.7. *Compact sets in $\mathbb{R}^n$ are closed and bounded.*

Proof. Consider a compact set $K \subset \mathbb{R}^n$. Let $x \in \mathbb{R}^n \setminus K$, and let $y \in K$. Since $x \neq y$, we choose open balls U_y around y and V_y around x such that $U_y \cap V_y = \emptyset$. Repeating this for all $y \in K$, we generate an open cover $\{U_y\}$ of K consisting of open balls. The compactness of K guarantees that this has a finite sub-cover, i.e. there is a finite set Y such that the collection $\{U_y\}_{y \in Y}$ covers X. As a result, the finite intersection of all V_y for $y \in Y$ is contained within $\mathbb{R}^n \setminus K$. Thus, x is in the exterior of K. Since x was chosen arbitrarily from $\mathbb{R}^n \setminus K$, we see that K is closed.

Now, consider the open cover $\{B_1(x)\}_{x \in K}$, and extract a finite sub-cover of unit open balls. The distance between any two points in K is at most the maximum distance between the centres of any two balls in our sub-cover, plus two. □

Lemma 1.8. *The intersection of a closed set and a compact set is compact.*

[1] If $x_1, x_2 \in [a,b]$ with $x_1 < x_2$, note that $a \leq x_1 < x_2 \leq b$, so

$$|x_2 - x_1| = x_2 - x_1 \leq b - a.$$

Proof. Let $F \subseteq \mathbb{R}^n$ be closed and let $K \subseteq \mathbb{R}^n$ be compact. Suppose that the open cover $\{U_\alpha\}$ of $F \cap K$ has no finite sub-cover. Now the complement $U = F^c$ is open in $\mathbb{R}^n$, hence the collection $\{U_\alpha\} \cup \{U\}$ is an open cover of K, and hence must admit a finite sub-cover of K. In particular, this must be a finite sub-cover of $F \cap K$. However, we can remove the set U from this sub-cover since it shares no element with $F \cap K$; as a result, our sub-cover must be a finite sub-collection of sets U_α, contradicting our assumption. This shows that $F \cap K$ is compact. □

Lemma 1.9 (Finite intersection property)**.** *Let $\{K_\alpha\}$ be a collection of compact sets in $\mathbb{R}^n$ which have the property that any finite intersection of them is non-empty. Then,*

$$\bigcap_\alpha K_\alpha \neq \emptyset.$$

Proof. Suppose to the contrary that the intersection of all K_α is empty. Fix an index β, and note that no element of K_β lies in every K_α. Set $J_\alpha = K_\alpha^c$, whence the collection $\{J_\alpha : \alpha \neq \beta\}$ is an open cover of K_β. This must admit a finite sub-cover $\{J_{\alpha_1}, \dots, J_{\alpha_k}\}$ of K_β. Thus, we must have

$$K_\beta^c \cup J_{\alpha_1} \cup \dots \cup J_{\alpha_k} = \mathbb{R}^n.$$

This immediately gives the contradiction

$$K_\beta \cap K_{\alpha_1} \cap \dots \cap K_{\alpha_k} = \emptyset.$$ □

Theorem 1.10 (Heine-Borel)**.** *Compact sets in $\mathbb{R}^n$ are precisely those that are closed and bounded.*

Proof. Given a compact set in $\mathbb{R}^n$, we have already shown that it must be closed and bounded. Next, if $F \subset \mathbb{R}^n$ is closed and bounded, it can be enclosed within a k-cell which we know is compact. Thus, F is the intersection of the closed set F and the compact k-cell, proving that F must be compact. □

1.5 Continuous maps

Definition 1.13. A map $f\colon X \to Y$ is continuous if the pre-image of every open set from Y is open in X.

Lemma 1.11. *A map $f\colon X \to Y$ is continuous if the pre-image of every closed set from Y is closed in X.*

Theorem 1.12. *The projection maps $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$, $\boldsymbol{x} \mapsto x_i$ are continuous.*

Proof. Let $U \subseteq \mathbb{R}$ be open; we claim that $\pi_i^{-1}(U)$ is open. Pick $\boldsymbol{x} \in \pi_i^{-1}(U)$, and note that $\pi_i(\boldsymbol{x}) = x_i \in U$. Thus, there exists $\delta > 0$ such that $(x_i - \delta, x_i + \delta) \subset U$. Now examine $B_\delta(\boldsymbol{x})$; for any point $\boldsymbol{y}$ within this open ball, we have $d(\boldsymbol{x}, \boldsymbol{y}) < \delta$ hence

$$|x_i - y_i|^2 \leq \sum_{k=1}^n (x_k - y_k)^2 = d(\boldsymbol{x}, \boldsymbol{y})^2 < \delta^2.$$

In other words, $\pi_i(\boldsymbol{y}) = y_i \in (x_i - \delta, x_i + \delta)$, hence $\pi_i\, B_\delta(\boldsymbol{x}) \subseteq (x_i - \delta, x_i + \delta) \subset U$. Thus, given arbitrary $\boldsymbol{x} \in \pi_i^{-1}(U)$, we have found an open ball $B_\delta(\boldsymbol{x}) \subset \pi_i^{-1}(U)$. $\square$

Lemma 1.13. *Finite sums, products, and compositions of continuous functions are continuous.*

Corollary 1.13.1. *A function $f\colon [a,b] \to \mathbb{R}^n$ is continuous if and only if the components, $\pi_i \circ f$, are continuous.*

Theorem 1.14. *All polynomial functions of the coordinates in $\mathbb{R}^n$ are continuous.*

Example. The unit sphere $S^{n-1} \subset \mathbb{R}^n$ is closed. It is by definition the pre-image of the singleton closed set $\{1\}$ under the continuous map

$$\boldsymbol{x} \mapsto x_1^2 + \cdots + x_n^2.$$

Theorem 1.15. *The continuous image of a compact set is compact.*

Proof. Let $f\colon X \to Y$ be continuous, where Y is the image of the compact set X, and let $\{U_\alpha\}$ be an open cover of Y. Then, the collection $\{f^{-1}(U_\alpha)\}$ is an open cover of X. Using the compactness of X, extract a finite sub-cover $f^{-1}(U_{\alpha_1}), \ldots, f^{-1}(U_{\alpha_k})$ of X. It follows that the collection $U_{\alpha_1}, \ldots, U_{\alpha_k}$ is a finite sub-cover of Y. $\square$

1.6 Connectedness

Definition 1.14. Let X be a topological space. A separation of X is a pair U, V of non-empty disjoint open subsets such that $X = U \cup V$.

Definition 1.15. A connected topological space is one which cannot be separated.

Lemma 1.16. *A topological space X is connected if and only if the only sets which are both open and closed are $\emptyset$ and X.*

Example. The intervals $(a,b) \subset \mathbb{R}$ are connected. To see this, suppose that U, V is a separation of (a,b). Pick $x \in U$, $y \in V$, and without loss of generality let $x < y$. Define $S = [x,y] \cap U$, and set $c = \sup S$. It can be argued that $c \in (a,b)$, but $c \notin U$, $c \notin V$, using the properties of the supremum.

Theorem 1.17. *The continuous image of a connected set is connected.*

Proof. Let f be a continuous map on the connected set X, and let Y be the image of X. If U, V is a separation of Y, then it can be shown that $\gamma^{-1}(U)$, $\gamma^{-1}(V)$ constitutes a separation of X, which is a contradiction. $\square$

Definition 1.16. A path γ joining two points $x, y \in X$ is a continuous map $\gamma\colon [a,b] \to X$ such that $\gamma(a) = x$, $\gamma(b) = y$.

Definition 1.17. A set in X is path connected if given any two distinct points in X, there exists a path joining them.

Lemma 1.18. *Every path connected set is connected.*

Proof. Let X be path connected, and suppose that U, V is a separation of X. Then, pick $x \in U$, $y \in V$, and choose a path $\gamma\colon [0,1] \to X$ between x and y. The sets $f^{-1}(U)$ and $f^{-1}(V)$ separate the interval $[0,1]$, which is a contradiction. $\square$

Example. All connected sets are not path connected. Consider the topologist's sine curve,

$$\left\{\left(x, \sin\frac{1}{x}\right) : 0 < x \leq 1\right\} \cup \{(0,0)\}.$$

Definition 1.18. The ϵ neighbourhood of a set K in a metric space X is defined as

$$\bigcup_{a \in K} B_\epsilon(a) = \bigcup_{a \in K} \{x \in X : d(x,a) < \epsilon\}.$$

Exercise 1.2. Let $K \subseteq \mathbb{R}^n$ be compact, and define $f\colon \mathbb{R}^n \to \mathbb{R}$,

$$f(x) = \operatorname{dist}(x, K) = \inf_{a \in K} d(x,a).$$

Show that f is continuous on $\mathbb{R}^n$, and $f^{-1}(\{0\}) = K$.

Exercise 1.3. If $K \subseteq \mathbb{R}^n$ is compact and $K \cap L = \emptyset$, then

$$\operatorname{dist}(K, L) = \inf_{a \in K} \operatorname{dist}(a, L) > 0.$$

Exercise 1.4. If $K \subseteq \mathbb{R}^n$ is compact and U is an open set containing K, then there exists $\epsilon > 0$ such that U contains the ϵ neighbourhood of K.

Is the compactness of K necessary?

2 Differential calculus

2.1 Differentiability

Definition 2.1. Let $f\colon (a,b) \to \mathbb{R}^n$, and let $f_i = \pi_i \circ f$ be its components. Then, f is differentiable at $t_0 \in (a,b)$ if the following limit exists.

$$f'(t_0) = \lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h}.$$

Remark. The vector $f'(t_0)$ represents the tangent to the curve f at the point $f(t_0)$. The full tangent line is the parametric curve $f(t) + f'(t_0)(t - t_0)$.

Definition 2.2. Let $U \subseteq \mathbb{R}^n$ be open, and let $f\colon U \to \mathbb{R}^m$. Then, f is differentiable at $x \in U$ if there exists a linear transformation $\lambda\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - \lambda h}{\|h\|} = 0.$$

The derivative of f at x is denoted by $\lambda = Df(x)$.

Remark. In a neighbourhood of x, we may approximate

$$f(x+h) \approx f(x) + Df(x)(h).$$

Remark. The statement that this quantity goes to zero means that each of the m components must also go to zero. For each of these limits, there are n axes along which we can let $h \to 0$. As a result, we obtain $m \times n$ limits, which allow us to identify the $m \times n$ components of the matrix representing the linear transformation λ (in the standard basis). These are the partial derivatives of f, and the matrix of λ is the Jacobian matrix of f evaluated at x.

Example. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. By choosing $\lambda = T$, we see that T is differentiable everywhere, with $DT(x) = T$ for every choice of $x \in \mathbb{R}^n$. This is made obvious by the fact that the best linear approximation of a linear map at some point is the map itself; indeed, the 'approximation' is exact.

Lemma 2.1. *If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \mathbb{R}^n$, with derivative $Df(x)$, then*

1. *f is continuous at x.*
2. *The linear transformation $Df(x)$ is unique.*

Proof. We prove the second part. Suppose that λ, μ satisfy the requirements for $Df(x)$; it can be shown that $\lim_{h\to 0}(\lambda-\mu)h/\|h\| = 0$. Now, if $\lambda v \neq \mu v$ for some non-zero vector $v \in \mathbb{R}^n$, then

$$\lambda v - \mu v = \frac{\lambda(tv) - \mu(tv)}{\|tv\|} \cdot \|v\| \to 0,$$

a contradiction. □

2.2 Chain rule

Exercise 2.1. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then, there exists $M > 0$ such that for all $\boldsymbol{x} \in \mathbb{R}^n$, we have

$$\|T\boldsymbol{x}\| \leq M\|\boldsymbol{x}\|.$$

Solution. Set $\boldsymbol{v}_i = T(\boldsymbol{e}_i)$ where $\boldsymbol{e}_i$ are the standard unit basis vectors of $\mathbb{R}^n$. Then,

$$\|T\boldsymbol{x}\| = \|\sum_i x_i\boldsymbol{v}_i\| \leq \sum_i \|x_i\boldsymbol{v}_i\| \leq \max\|\boldsymbol{v}_i\| \sum_i |x_i|.$$

Since each $|x_i| \leq \|\boldsymbol{x}\|$, set $M = n\max\|\boldsymbol{v}_i\|$ and write

$$\|T\boldsymbol{x}\| \leq \max\|\boldsymbol{v}_i\| \sum_i |x_i| \leq \max\|\boldsymbol{v}_i\| \cdot n\|\boldsymbol{x}\| = M\|\boldsymbol{x}\|.$$

Theorem 2.2. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $g\colon \mathbb{R}^m \to \mathbb{R}^k$ where f is differentiable at $a \in \mathbb{R}^n$ and g is differentiable at $f(a) \in \mathbb{R}^m$. Then, $g \circ f$ is differentiable, with $D(g\circ f)(a) = Dg(f(a)) \circ Df(a)$. Note that this means that the Jacobian matrices simply multiply.*

Proof. Set $b = f(a) \in \mathbb{R}^m$, $\lambda = Df(a)$, $\mu = Dg(f(a))$. Define

$$\begin{aligned} \varphi&\colon \mathbb{R}^n \to \mathbb{R}^m, & \varphi(x) &= f(x) - f(a) - \lambda(x-a), \\ \psi&\colon \mathbb{R}^m \to \mathbb{R}^k, & \psi(y) &= g(y) - g(b) - \mu(y-b). \end{aligned}$$

We claim that

$$\lim_{x\to a} \frac{g\circ f(x) - g\circ f(a) - \mu\circ\lambda(x-a)}{\|x-a\|} = 0.$$

Write the numerator as

$$g\circ f(x) - g\circ f(a) - \mu\circ\lambda(x-a) = \psi(f(x)) + \mu(\varphi(x)).$$

Note that

$$\lim_{x\to a}\frac{\varphi(x)}{\|x-a\|} = 0, \qquad \lim_{y\to b}\frac{\psi(y)}{\|y-b\|} = 0.$$

Thus, find $M > 0$ such that $\|\mu(\varphi(x))\| \leq \|\varphi(x)\|$ for all $x \in \mathbb{R}^n$, hence

$$\lim_{x\to a}\frac{\|\mu(\varphi(x))\|}{\|x-a\|} \leq \lim_{x\to a}\frac{M\|\varphi(x)\|}{\|x-a\|} = 0.$$

Now write

$$\lim_{f(x)\to b}\frac{\psi(f(x))}{\|f(x)-b\|} = 0,$$

hence for any $\epsilon > 0$, there is a neighbourhood of b on which

$$\|\psi(f(x))\| \leq \epsilon\|f(x) - b\| = \epsilon\|\varphi(x) + \lambda(x - a)\|.$$

Apply the triangle inequality and find $M' > 0$ such that

$$\|\psi(f(x))\| \leq \epsilon\|\varphi(x)\| + \epsilon M'\|x - a\|.$$

Thus,

$$\lim_{x \to a} \frac{\|\psi(f(x))\|}{\|x - a\|} \leq \lim_{x \to a} \frac{\epsilon\|\varphi(x)\|}{\|x - a\|} + \epsilon M' = \epsilon M'.$$

Since $\epsilon > 0$ was arbitrary, this limit is zero, completing the proof. □

2.3 Partial derivatives

Definition 2.3. Let $U \subseteq \mathbb{R}^n$ be open, and let $f\colon U \to \mathbb{R}$. The partial derivative of f with respect to the coordinate x_j at some $a \in U$ is defined by the following limit, if it exists.

$$\frac{\partial f}{\partial x_j}(a) = \lim_{h \to 0} \frac{f(a + h\boldsymbol{e}_j) - f(a)}{h}.$$

Lemma 2.3. *If $f\colon U \to \mathbb{R}$ is differentiable at a point $a \in \mathbb{R}^n$, then*

$$Df(a)(x_1, \ldots, x_n) = x_1 \frac{\partial f}{\partial x_1}(a) + \cdots + x_n \frac{\partial f}{\partial x_n}(a).$$

Example. Consider

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \qquad (x, y) \mapsto \begin{cases} xy/(x^2 + y^2), & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

Note that f is not differentiable at $(0, 0)$; it is not even continuous there. However, both partial derivatives of f exist at $(0, 0)$.

Lemma 2.4. *If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, then the matrix representation of $Df(a)$ in the standard basis is given by*

$$[Df(a)] = \left[\frac{\partial f_i}{\partial x_j}(a)\right]_{ij}.$$

Lemma 2.5. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $a \in \mathbb{R}^n$, and let $g\colon \mathbb{R}^m \to \mathbb{R}^k$ be differentiable at $f(a) \in \mathbb{R}^m$. Then, the matrix representation of $D(g \circ f)(a)$ in the standard basis is the product*

$$[D(g \circ f)(a)] = [Dg(f(a))][Df(a)] = \left[\sum_{\ell=1}^{m} \frac{\partial g_i}{\partial y_\ell} \frac{\partial f_\ell}{\partial x_j}\right]_{ij}.$$

In other words,

$$\frac{\partial}{\partial x_j}(g \circ f)_i(a) = \sum_{\ell=1}^{m} \frac{\partial g_i}{\partial y_\ell}(f(a)) \frac{\partial f_\ell}{\partial x_j}(a).$$

Example. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be differentiable, and let $\Gamma(f) = \{(x, y, f(x, y)) : x, y \in \mathbb{R}\}$ be the graph of f. Now, let $\gamma\colon [-1, 1] \to \Gamma(f)$ be a differentiable curve, represented by

$$\gamma(t) = (g(t), h(t), f(g(t), h(t))).$$

Then, we can compute the derivative

$$\gamma'(a) = \left(g'(a),\, h'(a),\, g'(a)\frac{\partial f}{\partial x} + h'(a)\frac{\partial f}{\partial y}\,\Big|_{(g(a),h(a))}\right)$$

Exercise 2.2. Consider the inner product map, $\langle \cdot, \cdot \rangle\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. What is its derivative?

Solution. We treat the inner product as a map $g\colon \mathbb{R}^{2n} \to \mathbb{R}$, which acts as

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle \cong g(x_1, \ldots, x_n, y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_n.$$

Now, note that

$$\frac{\partial g}{\partial x_i} = y_i, \qquad \frac{\partial g}{\partial y_i} = x_i.$$

Thus,

$$\begin{aligned} Dg(\boldsymbol{a}, \boldsymbol{b})(\boldsymbol{x}, \boldsymbol{y}) &= \sum_{i=1}^{n} x_i \frac{\partial g}{\partial x_i}(\boldsymbol{a}, \boldsymbol{b}) + \sum_{i=1}^{n} y_i \frac{\partial g}{\partial y_i}(\boldsymbol{a}, \boldsymbol{b}) \\ &= \sum_{i=1}^{n} x_i b_i + \sum_{i=1}^{n} y_i a_i \\ &= \langle \boldsymbol{x}, \boldsymbol{b} \rangle + \langle \boldsymbol{y}, \boldsymbol{a} \rangle. \end{aligned}$$

In other words, the matrix representation of the derivative of the inner product map at the point $(\boldsymbol{a}, \boldsymbol{b})$ is given by $[\boldsymbol{b}^\top\ \boldsymbol{a}^\top]$.

Exercise 2.3. Let $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ be a differentiable curve. What is the derivative of the real map $t \mapsto \|\gamma(t)\|^2$?

Solution. We write this map as $t \mapsto \langle \gamma(t), \gamma(t) \rangle$. Consider the scheme

$$\mathbb{R} \to \mathbb{R}^{2n} \to \mathbb{R}, \qquad t \mapsto \begin{bmatrix} \gamma(t) \\ \gamma(t) \end{bmatrix} \mapsto \langle \gamma(t), \gamma(t) \rangle.$$

Pick a point $t \in \mathbb{R}$, whence the derivative of the map at t is

$$\begin{bmatrix} \gamma(t)^\top & \gamma(t)^\top \end{bmatrix} \begin{bmatrix} \gamma'(t) \\ \gamma'(t) \end{bmatrix} = 2\langle \gamma(t), \gamma'(t) \rangle.$$

Remark. Consider the surface $S^{n-1} \subset \mathbb{R}^n$, and pick an arbitrary differentiable curve $\gamma\colon \mathbb{R} \to S^{n-1}$. Now, the tangent vector $\gamma'(t)$ is tangent to the sphere S^{n-1} at any point $\gamma(t)$. We claim that this tangent drawn at $\gamma(t)$ is always perpendicular to the position vector $\gamma(t)$. This is made trivial by our exercise: the map $t \mapsto \|\gamma(t)\|^2 = 1$ is a constant map since γ is a curve on the unit sphere. This means that it has zero derivative, forcing $\langle \gamma(t), \gamma'(t) \rangle = 0$.

2.3.1 Directional derivatives

Definition 2.4. Let $U \subseteq \mathbb{R}^n$ be open, and let $f\colon U \to \mathbb{R}$. The directional derivative of f along a direction $\boldsymbol{v} \in \mathbb{R}^n$ at a point $a \in U$ is defined by the following limit, if it exists.

$$\nabla_v f(a) = \lim_{h \to 0} \frac{f(a + h\boldsymbol{v}) - f(a)}{h}.$$

Example. Consider

$$f\colon \mathbb{R}^2 \to \mathbb{R}, \qquad (x, y) \mapsto \begin{cases} x^3/(x^2 + y^2), & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

Note that f is not differentiable at $(0, 0)$. However, all directional derivatives derivatives of f exist at $(0, 0)$. Indeed, consider a direction $(\cos\theta, \sin\theta)$, and examine the limit

$$\lim_{t \to 0} \frac{1}{t} \left[f(t\cos\theta, t\sin\theta) - f(0, 0) \right] = \cos^3\theta.$$

Definition 2.5. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be differentiable. The gradient of f is defined as the map

$$\nabla f\colon \mathbb{R}^n \to \mathbb{R}^n, \qquad x \mapsto \left[\frac{\partial f}{\partial x_i}(x) \right]_i.$$

Remark. The gradient at a point $x \in \mathbb{R}^n$ is thought of as a vector. In contrast, the derivative is thought of as a linear transformation. Otherwise, we see that $\nabla f(x) = [Df(x)]$.

Definition 2.6. Let $C^1(\mathbb{R}^n)$ be the set of real-valued differentiable functions on $\mathbb{R}^n$. Fix a point $a \in \mathbb{R}^n$, then fix a tangent vector $v \in \mathbb{R}^n$. Then, the map

$$\nabla_v \colon C^1(\mathbb{R}^n) \to \mathbb{R}, \qquad f \mapsto Df(a)(v)$$

is a linear functional. The quantity $\nabla_v f$ is called the directional derivative of f in the direction v at the point a.

Remark. We can represent ∇_v as the operator

$$\nabla_v(\cdot) = D(\cdot)(a)(v) = \sum_i v_i \frac{\partial}{\partial x_i}\Big|_a = v \cdot \nabla(\cdot).$$

Lemma 2.6. *The directional derivatives ∇_v form a vector space called the tangent space, attached to the point $a \in \mathbb{R}^n$. This can be identified with the vector space $\mathbb{R}^n$ by the natural map $\nabla_v \mapsto v$. The standard basis can be informally denoted by the vectors*

$$\nabla_{\boldsymbol{e}_1} \equiv \frac{\partial}{\partial x_1}, \ldots, \nabla_{\boldsymbol{e}_n} \equiv \frac{\partial}{\partial x_n}.$$

2.3.2 Differentiation on manifolds *

Definition 2.7. A homeomorphism is a continuous, bijective map whose inverse is also continuous.

Lemma 2.7. *Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be continuous. Denote the graph of f as*

$$\Gamma(f) = \{(x, f(x)) : x \in \mathbb{R}^n\}.$$

Then, $\Gamma(f)$ is a smooth manifold.

Proof. Consider the homeomorphism

$$\varphi \colon \Gamma(f) \to \mathbb{R}^n, \qquad (x, f(x)) \mapsto x.$$

This is clearly bijective, continuous (restriction of a projection map), with a continuous inverse (from the continuity of f). Call this homeomorphism φ a coordinate map on $\Gamma(f)$. □

Definition 2.8. Let $f \colon M \to \mathbb{R}$ where M is a smooth manifold, with a coordinate map $\varphi \colon M \to \mathbb{R}^n$. We say that f is differentiable at a point $a \in M$ if $f \circ \varphi^{-1} \colon \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\varphi(a)$.

Definition 2.9. Let $f\colon M \to \mathbb{R}$ where M is a smooth manifold, let $\varphi\colon M \to \mathbb{R}^n$ be a coordinate map, and let $a \in M$. Let $\gamma\colon \mathbb{R} \to M$ be a curve such that $\gamma(0) = a$, and further let γ be differentiable in the sense that $\varphi \circ \gamma\colon \mathbb{R} \to \mathbb{R}^n$ is differentiable. The directional derivative of f at a along γ is defined as

$$\frac{d}{dt} f(\gamma(t))\Big|_{t=0} = \lim_{h\to 0} \frac{f(\gamma(t+h)) - f(\gamma(t))}{h}\Big|_{t=0}.$$

Note that we are taking the derivative of $f \circ \gamma\colon \mathbb{R} \to \mathbb{R}$ in the conventional sense.

Lemma 2.8. *Let γ_1 and γ_2 be two curves in M such that $\gamma_1(0) = \gamma_2(0) = a$, and*

$$\frac{d}{dt}\varphi \circ \gamma_1(t)\Big|_{t=0} = \frac{d}{dt}\varphi \circ \gamma_2(t)\Big|_{t=0}.$$

In other words, γ_1 and γ_2 pass through the same point a at $t = 0$, and have the same velocities there. Then, the directional derivatives of f at a along γ_1 and γ_2 are the same.

Definition 2.10. Let M be a smooth manifold, and let $a \in M$. Consider the following equivalence relation on the set of all curves γ in M such that $\gamma(0) = a$.

$$\gamma_1 \sim \gamma_2 \quad \iff \quad \frac{d}{dt}\varphi \circ \gamma_1(t)\Big|_{t=0} = \frac{d}{dt}\varphi \circ \gamma_2(t)\Big|_{t=0}.$$

Each resultant equivalence class of curves is called a tangent vector at $a \in M$. Note that all these curves in a particular equivalence class pass through a with the same velocity vector.

The collection of all such tangent vectors, i.e. the space of all curves through a modulo the equivalence relation which identifies curves with the same velocity vector through a, is called the tangent space to M at a, denoted T_aM.

Remark. Each tangent vector $v \in T_aM$ acts on a differentiable function $f\colon M \to \mathbb{R}$ yielding a (well-defined) directional derivative at a.

$$v\colon C^1(M) \to \mathbb{R}, \qquad f \mapsto \frac{d}{dt} f(\gamma_v(t))\Big|_{t=0}.$$

Thus, the tangent space represents all the directions in which taking a derivative of f makes sense.

Remark. The tangent space T_aM is a vector space. Upon fixing f, the map $Df(a)\colon T_aM \to \mathbb{R}$, $v \mapsto vf(a)$ is a linear functional on the tangent space.

Remark. Given a tangent vector $v \in T_aM$, it can be identified with its corresponding velocity vector in $\mathbb{R}^n$. Thus, the tangent space T_aM can be identified with the geometric tangent plane drawn to the manifold M at the point a.

2.4 Mean value theorem

Consider a differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$, and fix $a \in \mathbb{R}^n$. Define the functions

$$g_i\colon \mathbb{R} \to \mathbb{R}, \qquad g_i(x) = f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n).$$

Then, each g_i is differentiable, with

$$g_i'(x) = \frac{\partial f}{\partial x_i}(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n).$$

By applying the Mean Value Theorem on some interval $[c, d]$, we can find $\alpha \in (c, d)$ such that $g_i(d) - g_i(c) = g_i'(\alpha)(d - c)$. In other words,

$$f(\ldots, d, \ldots) - f(\ldots, c, \ldots) = \frac{\partial f}{\partial x_i}(\ldots, \alpha, \ldots)(d - c).$$

Theorem 2.9. *Let $f \colon \mathbb{R}^n \to \mathbb{R}^m$ and $a \in \mathbb{R}^n$. Then, f is differentiable at a if all the partial derivatives $\partial f / \partial x_j$ exist in a neighbourhood of a and are continuous at a.*

Proof. Without loss of generality, let $m = 1$. We claim that

$$\lim_{h \to 0} \frac{1}{\|h\|} \| f(a + h) - f(a) - \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(a) h_i \| = 0.$$

Examine

$$\begin{aligned} f(a + h) - f(a) &= f(a_1 + h_1, \ldots, a_n + h_n) - f(a_1, \ldots, a_n) \\ &= f(a_1 + h_1, \ldots, a_n + h_n) - f(a_1 + h_1, \ldots, a_{n-1} + h_{n-1}, a_n) + \\ &\quad f(a_1 + h_1, \ldots, a_{n-1} + h_{n-1}, a_n) - f(a_1 + h_1, \ldots, a_{n-1}, a_n) + \\ &\quad \vdots \\ &\quad f(a_1 + h_1, a_2, \ldots, a_n) - f(a_1, \ldots, a_n) \\ &= \frac{\partial f}{\partial x_n}(c_n) h_n + \cdots + \frac{\partial f}{\partial x_1}(c_1) h_1. \end{aligned}$$

The last step follows from the Mean Value Theorem. As $h \to 0$, each $c_i \to a$. Thus,

$$\begin{aligned} \frac{1}{\|h\|} \| f(a + h) - f(a) - \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(a) h_i \| &= \frac{1}{\|h\|} \| \sum_{i=0}^{n} \left(\frac{\partial f}{\partial x_i}(c_i) - \frac{\partial f}{\partial x_i}(a) \right) h_i \| \\ &\leq \sum_{i=0}^{n} \left| \frac{\partial f}{\partial x_i}(c_i) - \frac{\partial f}{\partial x_i}(a) \right| \frac{|h_i|}{\|h\|} \\ &\leq \sum_{i=0}^{n} \left| \frac{\partial f}{\partial x_i}(c_i) - \frac{\partial f}{\partial x_i}(a) \right|. \end{aligned}$$

Taking the limit $h \to 0$, observe that $\partial f / \partial x_i(c_i) \to \partial f / \partial x_i(a)$ by the continuity of the partial derivatives, completing the proof. □

Corollary 2.9.1. *All polynomial functions on $\mathbb{R}^n$ are differentiable.*

Theorem 2.10. *Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be differentiable with continuous partial derivatives, and let $a \in \mathbb{R}^n$ be a point of local maximum. Then, $Df(a) = 0$.*

Proof. We need only show that each

$$\frac{\partial f}{\partial x_i}(a) = 0.$$

This must be true, since a is also a local maximum of each of the restrictions g_i as defined earlier. □

2.5 Inverse and implicit function theorems

Theorem 2.11 (Inverse function theorem). *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on a neighbourhood of $a \in \mathbb{R}^n$, and let $\det(Df(a)) \neq 0$. Then, there exist neighbourhoods U of a and W of $f(a)$ such that the restriction $f\colon U \to W$ is invertible. Furthermore, f^{-1} is continuous on U and differentiable on U.*

Lemma 2.12. *Consider a continuously differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$, and let M denote the surface defined by the zero set of f. Then, M can be represented as the graph of a differentiable function $h\colon \mathbb{R}^{n-1} \to \mathbb{R}$ at those points where $Df \neq 0$.*

Proof. Without loss of generality, suppose that $\partial f/\partial x_n \neq 0$ at some point $a \in M$. It can be shown that the map

$$F\colon \mathbb{R}^n \to \mathbb{R}^n, \qquad x \mapsto (x_1, x_2, \dots, x_{n-1}, f(x))$$

is invertible in a neighbourhood W of a, with a continuous and differentiable inverse of the form

$$G\colon \mathbb{R}^n \to \mathbb{R}^n, \qquad u \mapsto (u_1, u_2, \dots, u_{n-1}, g(u)).$$

Since $F \circ G$ must be the identity map on W, we demand

$$(x_1, x_2, \dots, x_{n-1}, f(x_1, x_2, \dots, x_{n-1}, g(x))) = (x_1, x_2, \dots, x_{n-1}, x_n).$$

Thus, the zero set of f in this neighbourhood of a satisfies $x_n = 0$, hence

$$f(x_1, x_2, \dots, x_{n-1}, g(x_1, x_2, \dots, x_{n-1}, 0)) = 0.$$

In other words, the part of the surface M in the neighbourhood of a is precisely the set of points

$$(x_1, x_2, \dots, x_{n-1}, g(x_1, x_2, \dots, x_{n-1}, 0)).$$

Simply set

$$h\colon \mathbb{R}^{n-1} \to \mathbb{R}, \qquad x \mapsto g(x_1, x_2, \dots, x_{n-1}, 0),$$

whence the surface M is locally represented by the graph of h. □

Remark. Note that by using

$$f(x_1, \dots, x_{n-1}, h(x_1, \dots, x_{n-1})) = 0$$

on the surface, we can use the chain rule to conclude that for all $1 \leq i < n$, we have

$$\frac{\partial f}{\partial x_i}(a) + \frac{\partial f}{\partial x_n}(a)\frac{\partial h}{\partial x_i}(a_1, \dots, a_{n-1}) = 0.$$

Theorem 2.13 (Implicit function theorem). *Let $f\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be continuously differentiable in an open set containing (a, b), with $f(a, b) = 0$. Let $\det(\partial f^j/\partial x_{n+k}(a, b)) \neq 0$. Then, there exists an open set $U \subset \mathbb{R}^n$ containing a, an open set $V \subset \mathbb{R}^m$ containing b, and a differentiable function $g\colon U \to V$ such that $f(x, g(x)) = 0$.*

Remark. The condition on the determinant can be rephrased as $\operatorname{rank} Df(a, b) = m$.

Theorem 2.14. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable, and let M be the surface defined by its zero set. Furthermore, let $\nabla f(a) \neq 0$ for some $a \in M$; thus, M can be locally represented by a graph on $\mathbb{R}^{n-1}$. Then, $\nabla f(a)$ is normal to the tangent vectors drawn at a to M; in fact, the perpendicular space of $\nabla f(a)$ is precisely the tangent space $T_a M$.*

Proof. Consider a tangent vector drawn at a to M, represented by the differentiable curve $\gamma\colon \mathbb{R} \to M$, $\gamma(0) = a$; note that we use the identification $\gamma'(0) = v \in \mathbb{R}^n$. Then, calculate

$$\frac{d}{dt} f(\gamma(t))\Big|_{t=0} = Df(\gamma(0))(\gamma'(0)) = Df(a)(v).$$

On the other hand, we have $f(\gamma(t)) = 0$ identically. Thus,

$$v \cdot \nabla f(a) = Df(a)(v) = 0.$$

□

2.6 Taylor's theorem

Theorem 2.15. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ have continuous second order partial derivatives. Then,*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Theorem 2.16. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ have continuous second order partial derivatives, and let $(x_0, y_0) \in \mathbb{R}^2$. Then, there exists $\epsilon > 0$ such that for all $\|(x - x_0, y - y_0)\| < \epsilon$,*

$$\begin{aligned} f(x,y) &= f(x_0, y_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) \\ &\quad + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x - x_0)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(y - y_0)^2 \\ &\quad + \frac{\partial^2 f}{\partial x \partial y}(x - x_0)(y - x_0) + R(x,y), \end{aligned}$$

where as $(x,y) \to (x_0, y_0)$, the remainder term vanishes as

$$\frac{|R(x,y)|}{\|(x - x_0, y - y_0)\|^2} \to 0.$$

All partial derivatives here are evaluated at (x_0, y_0).

Proof. This follows from applying the Taylor's Theorem in one variable to the real function $g\colon \mathbb{R} \to \mathbb{R}$, $t \mapsto f((1-t)(x_0, y_0) + t(x,y))$. □

2.7 Critical points and extrema

Definition 2.11. We say that $a \in \mathbb{R}^n$ is a critical point of $f\colon \mathbb{R}^n \to \mathbb{R}$ if all $\partial f / \partial x^j = 0$ there.

Lemma 2.17. *All points of extrema of a differentiable function are critical points.*

Proof. We already know that $Df(a) = 0$ where a is either a point of maximum or minimum. $\square$

Example. In order to find a point of extrema of a C^2-smooth function $f\colon \mathbb{R}^2 \to \mathbb{R}$, we first identify a critical point (x_0, y_0). Next, we must find a neighbourhood of (x_0, y_0) which contains no other critical points – to do this, apply Taylor's Theorem. Indeed, we see that

$$f(x,y) = f(x_0,y_0) + A(x-x_0)^2 + 2B(x-x_0)(y-y_0) + C(y-y_0)^2 + R_2.$$

For non-degeneracy of solutions, we demand $AC - B^2 \neq 0$, i.e. at (x_0, y_0), we want

$$\left[\frac{\partial^2 f}{\partial x \partial y}\right]^2 \neq \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}.$$

If $AC - B^2 > 0$ and $\partial^2 f/\partial x^2 > 0$, then we have found a point of minima; if $\partial^2 f/\partial x^2 < 0$, then we have found a point of maximum. If $AC - B^2 < 0$, then we have found a saddle point.

Example. Suppose that we wish to maximize the function $f\colon \mathbb{R}^2 \to \mathbb{R}$, given an equation of constraint $g = 0$, where $g\colon \mathbb{R}^2 \to \mathbb{R}$. Using the method of Lagrange multipliers, we look for solutions of the system

$$\begin{cases} \nabla f(x,y) + \lambda \nabla g(x,y) = 0, \\ g(x,y) = 0. \end{cases}$$

3 Integral calculus

3.1 Path integrals

Definition 3.1. A closed curve $\gamma\colon [a,b] \to \mathbb{R}^n$ is closed if $\gamma(a) = \gamma(b)$. It is called simple if it has no self intersections.

Definition 3.2. Let $p, q\colon U \to \mathbb{R}$ be continuous, where $U \subseteq \mathbb{R}^2$ is an open set, and let $\gamma\colon [a,b] \to U$ be piecewise smooth, i.e. smooth on (a,b) at all but finitely many points. Then, we define

$$\int_\gamma p\,dx + q\,dy = \int_a^b p(\gamma(t))\,\gamma_1'(t) + q(\gamma(t))\,\gamma_2'(t)\,dt.$$

Lemma 3.1. *Let $\gamma\colon [a,b] \to \mathbb{R}^2$ be a smooth curve, and let $\varphi\colon [c,d] \to [a,b]$ be smooth, such that $\varphi(c) = a$ and $\varphi(d) = b$. Then, the composition $\gamma\colon \varphi\colon [c,d] \to \mathbb{R}^2$ is a smooth curve, and*

$$\int_{\gamma\circ\varphi} p\,dx + q\,dy = \int_c^d \left[p(\gamma\circ\varphi(s))\,\gamma_1'(\varphi(s)) + p(\gamma\circ\varphi(s))\,\gamma_2'(\varphi(s))\right]\varphi'(s)\,ds.$$

By substituting the parameter $\varphi(s) = t$, $\varphi'(s)\,ds = dt$, we retrieve

$$\int_{\gamma\circ\varphi} p\,dx + q\,dy = \int_a^b p(\gamma(t))\,\gamma_1'(t) + q(\gamma(t))\,\gamma_2'(t)\,dt = \int_\gamma p\,dx + q\,dy.$$

Theorem 3.2. *Let $p, q\colon U \to \mathbb{R}^2$ be continuous, and let $\gamma\colon [a,b] \to U$ be a smooth curve. The integral*

$$\int_\gamma p\,dx + q\,dy$$

depends only on the endpoints of γ if and only if there exists $u\colon U \to \mathbb{R}$ such that

$$p = \frac{\partial u}{\partial x}, \qquad q = \frac{\partial u}{\partial y}.$$

In other words, we demand that that the vector field (p,q) be the gradient of u.

Proof. First suppose that there exists u such that $\nabla u = (p,q)$. Then,

$$\int_\gamma p\,dx + q\,dy = \int_a^b \frac{\partial u}{\partial x}(\gamma(t))\gamma_1'(t) + \frac{\partial u}{\partial y}(\gamma(t))\gamma_2'(t)\,dt.$$

The chain rule shows that this is simply

$$\int_a^b \frac{d}{dt}u(\gamma(t))\,dt = u(\gamma(b)) - u(\gamma(a)).$$

Conversely, suppose that the given integral depends only on the endpoints of γ. Given two points $\alpha, \beta \in U$, we construct a path from α to β by travelling only along the axes. Pick $(x,y) \in U$, and define $u\colon U \to \mathbb{R}$,

$$u(x,y) = \int_\gamma p\,dx + q\,dy,$$

where γ is such a polygonal path from a fixed point α to (x,y). Note that u is well-defined by the independence of choice of path γ. □

Example. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be continuous, and let γ be a smooth curve in $\mathbb{R}^2$. We may denote

$$\int_\gamma f\cdot ds = \int_\gamma f^1\,dx + f^2\,dy.$$

3.2 Multiple integrals

Definition 3.3. Let $f\colon [a_1, b_1] \times [a_2, b_2] \to \mathbb{R}$ be continuous. Now, let P be a partition of the rectangular domain into $n \times n$ sub-rectangles, and define

$$M_{ij} = \sup_{[x_i,x_{i+1}]\times[y_j,y_{j+1}]} f(x, y), \qquad y_{ij} = \inf_{[x_i,x_{i+1}]\times[y_j,y_{j+1}]} f(x, y).$$

We also define,

$$U(f, P) = \sum_{i,j} M_{ij}(x_i - x_{i-1})(y_j - y_{j-1}), \qquad L(f, P) = \sum_{i,j} m_{ij}(x_i - x_{i-1})(y_j - y_{j-1}).$$

Finally define the upper and lower sums

$$U(f) = \sup_P U(f, P), \qquad L(f) = \sup_P L(f, P).$$

Then, f is Riemann integrable if $U(f) = L(f)$, and this integral is denoted by

$$\int_{[a_1,b_1]\times[a_2,b_2]} f$$

Remark. This definition naturally extends to integrals over any k-cell.

Definition 3.4. A measure zero set $E \subset \mathbb{R}^n$ is such that given any $\epsilon > 0$, there exists a countable collection of rectangles $\{A_j\}$ such that their union contains E, and the sum of their volumes is less than ϵ.

Example. Any countable subset of $\mathbb{R}^n$ has measure zero.

Example. Any line in $\mathbb{R}^2$, plane in $\mathbb{R}^3$, etc. has measure zero.

Lemma 3.3. *The countable union of measure zero sets has measure zero.*

Theorem 3.4. *A bounded function $f\colon A \to \mathbb{R}$ where $A \subset \mathbb{R}^n$ is a rectangle is integrable if and only if for every $\epsilon > 0$, there exists a partition P of A such that $U(f, P) - L(f, P) < \epsilon$.*

Theorem 3.5. *Let $f\colon A \to \mathbb{R}$ be bounded, where $A \subset \mathbb{R}^n$ is a rectangle. Then, f is Riemann integrable if and only if its set of discontinuities has measure zero.*

Theorem 3.6. *Let $f\colon A_1 \times A_2 \to \mathbb{R}$ be continuous, where $A_1 \subset \mathbb{R}^n$ and $A_2 \subset \mathbb{R}^m$ are closed rectangles. Then, we can write*

$$\int_{A_1\times A_2} f = \int_{A_2} \left(\int_{A_1} f(x,y)\,dx \right) dy = \int_{A_1} \left(\int_{A_2} f(x,y)\,dy \right) dx.$$

Theorem 3.7 (Green's theorem)**.** *Let γ be a smooth simple closed curve in $\mathbb{R}^2$ oriented counter-clockwise, and let Ω be the region enclosed by γ. If $p, q\colon \Omega \to \mathbb{R}$ have continuous partial derivatives, then*

$$\int_\gamma p\,dx + q\,dy = \iint_\Omega \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) dx\,dy.$$

Example. Let $\gamma\colon [0, 2\pi] \to \mathbb{R}$, $\gamma(t) = (\cos t, \sin t)$. Then, γ enclosed the unit disc in $\mathbb{R}^2$. To calculate its area, we can set $p = 0$, $q = x$, giving

$$\iint_\Omega dx\,dy = \int_0^{2\pi} \cos^2 t\,dt = \pi.$$

Another option is to set $p = -y/2$, $q = x/2$, giving

$$\iint_\Omega dx\,dy = \frac{1}{2}\int_0^{2\pi} \cos^2 t + \sin^2 t\,dt = \frac{1}{2}\int_0^{2\pi} dt = \pi.$$

Theorem 3.8 (Change of variables)**.** *Let $U \subset \mathbb{R}^n$ be an open set, and get $g\colon U \to \mathbb{R}^n$ be a bijective, continuously differentiable map such that $\det Dg(x) \neq 0$ on U. If $f\colon g(U) \to \mathbb{R}$ is integrable, then*

$$\int_{g(U)} f = \int_U f \circ g\,|\det Dg|$$

Definition 3.5. For $S \subset A \subset \mathbb{R}^n$ where A is a rectangle, we can define

$$\int_S f = \int_A f \cdot \chi_S.$$

Here, χ_S is the characteristic function of S, defined as

$$\chi_S(x) = \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{if } x \notin S. \end{cases}$$

Lemma 3.9. *For any set $A \subset \mathbb{R}^n$, the characteristic function χ_A is continuous precisely on $\mathbb{R}^n \setminus \partial A$, and discontinuous on ∂A.*

Definition 3.6. Let $A \subset \mathbb{R}^n$, and let $\{U_\alpha\}$ be an open cover of A. Suppose that $\{\psi_\alpha\}$ are smooth functions defined on a neighbourhood of A, satisfying the following properties.

1. $0 \leq \psi_\alpha \leq 1$ on A.
2. For each $x \in A$, there exists a neighbourhood U_x of x such that only finitely many ψ_α are non-zero on U_x.
3. $\sum_\alpha \psi_\alpha = 1$ on A.
4. Each support of ψ_α is contained in U_α.

Remark. The support of a function is the closure of the set on which it is non-zero.

The collection $\{\psi_\alpha\}$ is called a partition of unity subordinate to the open cover $\{U_\alpha\}$.

Theorem 3.10. *For every set $A \subset \mathbb{R}^n$ and every locally finite open cover $\{U_\alpha\}$ of A, there exists a partition of unity subordinate to that open cover.*