MA2202: Probability I

# Markov chains

Spring 2021

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**Definition 5.1** (Markov chain)**.** Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of random variables taking values in $\{0, 1, \ldots, N\}$ such that

$$P(X_{n+1} = s_{n+1} | X_n = s_n, \ldots, X_0 = s_0) = P(X_{n+1} = s_{n+1} | X_n = s_n).$$

Then, the sequence $\{X_n\}$ is a Markov chain.

**Definition 5.2** (Transition probabilities)**.** We define

$$p_{ij} = P(X_{n+1} = j \mid X_n = i).$$

If $p_{ij}$ does not depend on $n$, then it is called a stationary transition probability from the $i^{\text{th}}$ state to the $j^{\text{th}}$ state.

*Remark.* Note that

$$\sum_{j=0}^{N} P(X_{n+1} = j \mid X_n = i) = \sum_{j=0}^{N} p_{ij} = 1.$$

**Definition 5.3** (Stochastic matrix)**.** The stochastic matrix or transition probability matrix $\mathbb{P}$ is defined such that $\mathbb{P}_{ij} = p_{ij}$, where $0 \leq i, j \leq N$.

*Remark.* Note that setting $\mathbf{1} = (1 \ldots 1)^{\top}$, we have $\mathbb{P}\mathbf{1} = \mathbf{1}$.

**Lemma 5.1.** *For a Markov chain with stationary transition probabilities,*

$$P(X_n = s_n, \ldots, X_0 = s_0) = p_{s_{n-1}s_n} p_{s_{n-2}s_{n-1}} \ldots p_{s_0 s_1} P(X_0 = s_0).$$

**Definition 5.4** ($n$-stage transition probability matrix)**.** We define $\mathbb{P}^{(n)}$ such that

$$\mathbb{P}_{ij}^{(n)} = P(X_n = j \mid X_0 = i).$$

Note that for stationary transition probabilities, this is equivalent to

$$\mathbb{P}_{ij}^{(n)} = P(X_{m+n} = j \mid X_m = i).$$

**Lemma 5.2.** *The $n$-stage transition matrices are simply the powers of the stochastic matrix,*

$$\mathbb{P}^{(n)} = \mathbb{P}^n.$$

*Proof.* Note that

$$P(X_n = s_n | X_0 = s_0) = \sum_{s_{n-1},\ldots,s_1} p_{s_{n-1}s_n} \ldots p_{s_0 s_1}.$$

□

**Lemma 5.3.** *The probability mass function of $X_n$ is given by*

$$P(X_n = j) = \sum_{i=0}^{N} P(X_n = j \mid X_0 = i)\, P(X_0 = i) = \sum_{i=0}^{N} \mathbb{P}_{ij}^{(n)}\, P(X_0 = i).$$

**Theorem 5.4** (Chapman-Kolmogorov equation)**.**

$$\mathbb{P}_{ij}^{(n)} = \left[\mathbb{P}^{(n-r)}\mathbb{P}^{(r)}\right]_{ij} = \sum_{k=0}^{N} \mathbb{P}_{ik}^{(n-r)}\mathbb{P}_{kj}^{(r)}.$$