MA2202: Probability I

Introduction to probability

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Satvik Saha
19MS154

*Indian Institute of Science Education and Research, Kolkata,
Mohanpur, West Bengal, 741246, India.*

Definition 1.1 (Experiment)**.** An experiment is an act which can be repeated under similar conditions.

Example. Tossing a fair coin constitutes an experiment. Here, the possible outcomes of the experiment are 'heads' or 'tails'.

Definition 1.2 (Random experiment)**.** A random experiment is one where there is more than one possible outcome, and the outcome of the experiment cannot be determined beforehand.

Example. A coin toss, or the roll of a die is typically regarded as a random experiment.

Definition 1.3 (Sample space)**.** A sample space Ω is the set of all outcomes of an experiment.

Example. The sample space of rolls of a single die is $\Omega = \{1, 2, 3, 4, 5, 6\}$. Note that this is a finite, discrete sample space.

Example. In a game of guessing a particular natural number, the sample space is the set of all natural numbers $\mathbb{N}$. Note that this is an infinite, discrete sample space.

Example. The temperature in a room may vary continuously. Thus, the sample space of temperatures is a continuous sample space.

Definition 1.4 (Events)**.** A set of events $\mathcal{E}$ is a collection of measurable subsets of a sample space such that $\Omega \in \mathcal{E}$, it is closed under complementing, and it is closed under countable unions.

Remark. Formally, the event space $\mathcal{E} \subseteq \mathcal{P}(\Omega)$ forms a σ-algebra. The pair $(\Omega, \mathcal{E})$ is called a measurable space.

Example. We may have $\mathcal{E} = \{\emptyset, \{2,4,6\}, \{1,3,5\}, \Omega\}$ as our set of events in the case of rolling a die. Obtaining an even number is an event.

Note that the set of events is also closed under countable intersections, because for a countable set of events $\{E_n\}_n$, we have

$$\bigcap_{n=1}^{\infty} E_n = \left(\bigcup_{n=1}^{\infty} E_n^c\right)^c$$

by De Morgan's Law, and $E_n^c \in \mathcal{E}$.

Definition 1.5 (Probability)**.** A probability measure is a function $P \colon \mathcal{E} \to [0,1]$ such that $P(\emptyset) = 0$, $P(\Omega) = 1$, and for any countable collection of pairwise disjoint events $\{E_n\}_n$, we have

$$P(E) = \sum_{n=1}^{\infty} P(E_n), \qquad E = \bigcup_{n=1}^{\infty} E_n.$$

Note that we obtain the relation

$$P(A^c) = 1 - P(A)$$

directly by noting that $A \cup A^c = \Omega$ and $P(\Omega) = 1$.

Definition 1.6 (Probability space)**.** A probability space $(\Omega, \mathcal{E}, P)$ consists of a sample space Ω together with a set of events $\mathcal{E}$ and a probability measure P.

Example. In the context of a coin toss, set $\Omega = \{H, T\}$, $\mathcal{E} = \{\emptyset, \{H\}, \{T\}, \{H,T\}\}$ and define $P \colon \mathcal{E} \to [0,1]$ such that $P(H) = P(T) = 1/2$. It can be verified that $\mathcal{E}$ is a σ-algebra and that P is a probability measure, so the triple $(\Omega, \mathcal{E}, P)$ is indeed a probability space.

Definition 1.7 (Equally likely events)**.** Two events $A, B \in \mathcal{E}$ are said to be equally likely if $P(A) = P(B)$.

The classical definition of probability states that if the sample space Ω consists of N equally likely events, then the probability of an event $E \in \mathcal{E}$ is given by

$$P(E) = \frac{|E|}{N}.$$

Note that this assumes that the notion of equally likely events is known beforehand.

The frequency definition of probability involves performing an experiment n times, denoting $f_n(E)$ as the frequency of the event E over these iterations, and defining

$$P(E) = \lim_{n\to\infty} \frac{f_n(E)}{n}.$$

Note that such a limit may not always be well defined.

Definition 1.8 (Mutually exclusive events)**.** Two events $A, B \in \mathcal{E}$ are called mutually exclusive if $A \cap B = \emptyset$.

Definition 1.9 (Exhaustive events)**.** A set of events $S \subseteq \mathcal{E}$ is called exhaustive if

$$\Omega = \bigcup_{E\in S} E.$$

Example. For any event $A \in \mathcal{E}$, we see that A and A^c are mutually exclusive and exhaustive.

Theorem 1.1 (Principle of Inclusion and Exclusion)**.** *For events $A_1, A_2, \dots, A_n \in \mathcal{E}$, we have*

$$\begin{aligned} P(A_1 \cup \dots \cup A_n) &= \sum P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \\ &\sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \dots + (-1)^{n-1} P(A_1 \cap \dots A_n). \end{aligned}$$

Proof. This follows by induction. The base case of $n = 2$ states

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2),$$

which follows form the fact that the sets $A_1 \setminus A_2$, $A_1 \cap A_2$ and $A_2 \setminus A_1$ are pairwise disjoint.

For the induction step, assume that the expansion holds for $n = m \geq 2$ and note that

$$P\left(A_{m+1} \cap \bigcup_{i=1}^{m} A_i\right) = P\left(\bigcup_{i=1}^{m} A_i \cap A_{m+1}\right).$$

Putting the $n = m + 1$ case into the $n = 2$ case and expanding the above $n = m$ case, the full expansion will follow. □

Theorem 1.2 (Boole's inequality)**.** *For events $A_1, A_2, \dots, A_n \in \mathcal{E}$, we have*

$$P(A_1 \cup \dots A_n) \leq \sum P(A_i).$$

Proof. This is clearly true for $n = 2$, since

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq P(A_1) + P(A_2).$$

Define

$$B_i = A_i \setminus \bigcup_{j=1}^{i-1} A_j.$$

Note that $\cup B_i = \cup A_i$, and all B_i are pairwise disjoint. In addition, $B_i \subseteq A_i$, so $P(B_i) \leq P(A_i)$. Thus,

$$P(A_1 \cup \cdots \cup A_2) \ = \ \sum P(B_i) \ \leq \ \sum P(A_i).$$ □

Theorem 1.3 (Bonferroni's inequality)**.** *For events $A_1, A_2, \ldots, A_n \in \mathcal{E}$, we have*

$$P(A_1 \cap \cdots \cap A_n) \geq \sum P(A_i) - (n-1).$$

Proof. This holds for $n = 2$, since

$$P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2) \geq P(A_1) + P(A_2) - 1.$$

For the induction step, suppose this holds for $n = m \geq 2$. Thus,

$$P(A_1 \cap \cdots \cap A_m \cap A_{m+1}) \geq P(A_1 \cap \cdots \cap A_m) + P(A_m) - 1 \geq \sum P(A_i) - m.$$ □