MA 2202 : Probability I

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Exercise 1 Let X, Y and Z be independent identically distributed positive random variables. Determine

$$\mathrm{E}\left[\frac{2X+3Y}{X+Y+Z}\right].$$

Solution Using the fact that X, Y, Z are identically distributed, we use a symmetry argument and permute them cyclically, which gives

$$\mathrm{E}\left[\frac{2X+3Y}{X+Y+Z}\right] = \mathrm{E}\left[\frac{2Y+3Z}{X+Y+Z}\right] = \mathrm{E}\left[\frac{2Z+3X}{X+Y+Z}\right].$$

Adding these up and using linearity of expectation, we have

$$3\,\mathrm{E}\left[\frac{2X+3Y}{X+Y+Z}\right] = \mathrm{E}\left[\frac{2(X+Y+Z)+3(Y+Z+X)}{X+Y+Z}\right] = \mathrm{E}\left[5\right] = 5.$$

Hence,

$$\mathrm{E}\left[\frac{2X+3Y}{X+Y+Z}\right] = \frac{5}{3}.$$

Exercise 2 A gambler plays a game in which on each play he wins Rs. 1 with probability p and loses Rs. 1 with probability $q = 1 - p$. Once he loses all his money, he cannot gamble any more. The gambler starts with Rs. R, where $R \in \mathbb{N}$.

(i) To evade the possibility of losing all his money, he chooses an integer $M > R$ and decides to quit gambling as soon as he has Rs. M. Find the probability that he loses all his money if

   (a) $p \neq q$.

   (b) $p = 1 = 1/2$.

(ii) Show that if $p \leq q$ and if instead of setting a target M to quit gambling, the gambler keeps on gambling irrespective of his gains or losses, then eventually he will lose all his money.

Solution Let $\{X_n\}_{n=1}^{\infty}$ be a Markov chain where X_n denotes the money the gambler has after gambling n times (we fix $X_0 = R$ for notational purposes). Recall that the Markov property is indeed satisfied – the money the gambler has at stage $n+1$ depends only on how much he had at stage n, and whether he wins or loses.

(i) The gambler must either win or lose money at each round. Say he currently has money $X_n = k$. If $1 < k < M - 1$, he could have reached this point by winning Rs. 1 from $X_{n-1} = k - 1$, or by losing Rs. 1 from $X_{n-1} = k + 1$.

   Let P_k be the probability that the gambler loses all his money given that he currently has Rs. k. If $0 < k < M$, note that

$$P_k = pP_{k+1} + qP_{k-1}.$$

   This is because the gambler wins on the next turn with probability p, getting Rs. $k+1$. Now, the Markov property means that his odds of ruin henceforth only depend on the current state, which we have defined as P_{k+1}. The same applies for the qP_{k-1} term. Also, since the game terminates whenever $k = 0$ or $k = M$

$$P_0 = 1, \qquad P_M = 0.$$

Note that $R \neq 0, M$. Now, write

$$P_k = (p+q)P_k = pP_{k+1} + qP_{k-1}, \qquad p(P_{k+1} - P_k) = q(P_k - P_{k-1}).$$

Therefore, using $P_0 = 1$,

$$P_{k+1} - P_k = \frac{q}{p}(P_k - P_{k-1}) = \cdots = \frac{q^k}{p^k}(P_1 - 1).$$

Now by telescoping the series,

$$P_{k+1} - 1 = (P_{k+1} - P_k) + (P_k - P_{k-1}) + \cdots + (P_1 - 1) = \sum_{n=0}^{k} \frac{q^n}{p^n}(P_1 - 1).$$

This is a geometric series, which for $p \neq q$ is

$$P_{k+1} - 1 = \frac{1 - (q/p)^{k+1}}{1 - (q/p)}(P_1 - 1)$$

and for $p = q$ is

$$P_{k+1} - 1 = (k+1)(P_1 - 1).$$

To evaluate $1 - P_1$, set $k + 1 = M$. Demanding $P_M = 0$ gives

$$1 - P_1 = \begin{cases} \frac{1-(q/p)}{1-(q/p)^M}, & \text{if } p \neq q, \\ \frac{1}{M}, & \text{if } p = q = 1/2. \end{cases}$$

Thus, setting $k + 1 = R$, we have the desired probability of ruin

$$P_R = \begin{cases} 1 - \frac{1-(q/p)^R}{1-(q/p)^M}, & \text{if } p \neq q, \\ 1 - \frac{R}{M}, & \text{if } p = q = 1/2. \end{cases}$$

(ii) Note that when $p \leq q$, setting $1 < \alpha = q/p$, we have

$$\frac{1 - \alpha^R}{1 - \alpha^M} = \frac{\alpha^{-M} - \alpha^{R-M}}{\alpha^{-M} - 1}.$$

Since $\alpha > 1$, we have $\alpha^{-M} \to 0$ as $M \to \infty$. Also, $\alpha^{R-M} \to 0$ since we always have $R < M$. Thus,

$$1 - \frac{1 - \alpha^R}{1 - \alpha^M} \to 1$$

as $M \to \infty$. Similarly, we have $R/M \to 0$, so

$$1 - \frac{R}{M} \to 1.$$

In any case, $P_R \to 1$ as $M \to \infty$. This means that if the gambler keeps playing irrespective of gains or losses, he will be eventually lose all his money (ruin is a probability one event).