MA 2202 : Probability I

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Exercise 1 Let X and Y be jointly distributed continuous random variables. Let the marginal distribution of Y be standard normal. If

$$\mathrm{E}[X|Y] = 10Y^4 - 2Y^3 + 5Y^2 - 4Y + 3,$$

determine $\mathrm{E}[X]$.

Solution We use the formula

$$\mathrm{E}[\mathrm{E}[X|Y]] = \mathrm{E}[X]$$

to write

$$\mathrm{E}[X] = 10\,\mathrm{E}[Y^4] - 2\,\mathrm{E}[Y^3] + 5\,\mathrm{E}[Y^2] - 4\,\mathrm{E}[Y] + 3.$$

Now for a standard normal distribution, recall that the moments are given by

$$\mathrm{E}[Y^{2n-1}] = 0, \qquad \mathrm{E}[Y^{2n}] = (2n-1)!!,$$

so the required expectation is

$$\mathrm{E}[X] = 10 \cdot 3 \cdot 1 + 5 \cdot 1 + 3 = 38.$$

Exercise 2 Let X and Y be independent identically distributed Poisson(λ) random variables. Let $Z = \min(X, Y)$ and $W = \max(X, Y)$.

(a) Determine the correlation coefficient of $Z + W$ and $X - Y$.

(b) Write down the joint probability mass function of Z and W.

(c) Compute $\mathrm{E}[Z|W]$.

Solution

(a) Note that $X + Y = Z + W$, since the sum of the maximum and minimum is always the sum of the two. Recall that the probability mass function of a Poisson random variable

$$p_X(x) = p_Y(x) = \frac{\lambda^n}{n!} e^{-\lambda},$$

for $n \geq 0$, and $\mathrm{E}[X] = \mathrm{E}[Y] = \lambda$. Therefore,

$$\mathrm{E}[X + Y] = 2\lambda, \qquad \mathrm{E}[X - Y] = 0.$$

Also,

$$\mathrm{Cov}[X + Y, X - Y] = \mathrm{E}[(X + Y)(X - Y)] - \mathrm{E}[X + Y]\,\mathrm{E}[X - Y] = \mathrm{E}[X^2 - Y^2] = 0.$$

Thus, the correlation coefficient is simply

$$\rho_{X+Y, X-Y} = \frac{\mathrm{Cov}[X + Y, X - Y]}{\sigma_{X+Y}\sigma_{X-Y}} = 0.$$

(b) Recall that since X and Y are independent,

$$P(X = m, Y = n) = P(X = n, Y = m) = p_X(m)p_Y(n) = \frac{\lambda^{m+n}}{m!n!} e^{-2\lambda}.$$

We have also shown that

$$p_{Z,W}(m,n) = \begin{cases} P(X=m, Y=n), & \text{if } 0 \le m \le n, \\ 0, & \text{otherwise.} \end{cases}$$

This is simply because $Z \le W$. Now, if $0 \le m < n$, then

$$p_{Z,W}(m,n) = P(X=m, Y=n) + P(X=n, Y=m) = \frac{2\lambda^{m+n}}{m!n!} e^{-2\lambda}.$$

If $m = n$, then simply

$$p_{Z,W}(m,m) = P(X=m, Y=m) = \frac{\lambda^{2m}}{(m!)^2} e^{-2\lambda}.$$

Thus,

$$p_{Z,W}(m,n) = \begin{cases} \frac{2\lambda^{m+n}}{m!n!} e^{-2\lambda}, & \text{if } 0 \le m < n, \\ \frac{\lambda^{2m}}{(m!)^2} e^{-2\lambda}, & \text{if } 0 \le m = n, \\ 0, & \text{otherwise.} \end{cases}$$

(c) Set $V = \mathrm{E}[Z|W]$. Note that

$$P(Z=m|W=n) = \frac{P(Z=m, W=n)}{P(W=n)}.$$

We thus compute the probability mass function of W. Now,

$$P(W \le n) = P(X \le n, Y \le n) = P(X \le n)^2.$$

This gives,

$$\begin{aligned} P(W=n) &= P(W \le n) - P(W \le n-1) \\ &= P(X \le n)^2 - P(X \le n-1)^2 \\ &= [P(X \le n) + P(X \le n-1)] \cdot P(X=n). \end{aligned}$$