MA 2202 : Probability I

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Exercise 1 Let X and Y be two independent identically distributed random variables taking values in $\{0, 1, \dots, n\}$. Determine the probability distribution of $Z = X + Y$.

Solution Let X and Y share the probability mass functions $p_X = p_Y = p$. We seek p_Z, where

$$p_Z(z) = P(Z = z) = P(X + Y = z) = \sum_{s=0}^{z} P(X = s, Y = z - s) = \sum_{s=0}^{z} p(s)p(z - s).$$

Here, we have summed over all possible ways of splitting z among the two variables X and Y, and have used the independence of the variables to split the joint distribution as a product $p_{X,Y}(x, y) = p_X(x)p_Y(y)$.

Now, note that if $z < 0$ or $z > 2n$, we immediately get $p_Z(z) = 0$. This is because there are no terms in our sum, i.e. no ways of adding integers in the given range to get such a z. Similarly, if $n < z \leq 2n$, our sum only goes as far as $s = n$ since subsequent terms are cut off by $p(s)$ being zero. Also, we demand $z - s \leq n$, so $s \geq z - n$. In other words, the sum is over $s = z - n, \dots, n$. If $0 \leq z \leq n$, all the terms in our sum survive.

Putting all this together, suppose that the common probability mass function is uniform, with $p(x) = 1/(n + 1) = q$ when $0 \leq x \leq n$ and $p(x) = 0$ otherwise. Then we have

$$p_Z(z) = \begin{cases} 0, & \text{if } z < 0, \\ (z + 1)q^2, & \text{if } 0 \leq z \leq n, \\ (2n - z + 1)q^2, & \text{if } n \leq z \leq 2n, \\ 0, & \text{if } z > 2n. \end{cases}$$

If we plot such a function, we see that $p_Z(0) = q^2$, which increases with z by one each time until we get a peak $p_Z(n) = (n + 1)q^2$, then it falls off symmetrically until $p_Z(2n) = q^2$. The total 'area' covered is can be calculated by joining the triangular halves into a square with side $n + 1$, hence we have the sum

$$\sum_{z=0}^{2n} p_Z(z) = (n + 1)^2 q^2 = 1.$$

Exercise 2 Let X and Y be two independent identically distributed continuous Uniform$(0, 1)$ random variables taking values in $\{0, 1, \dots, n\}$. Determine the probability distribution of $Z = X + Y$.

Solution Let X and Y share the common probability density function $f_X = f_Y = f$. We seek f_Z, where

$$f_Z(z) = P(X + Y = z) = \int_{\mathbb{R}} f_{X,Y}(X = t, Y = z - t)\, dt = \int_{\mathbb{R}} f(t)f(z - t)\, dt.$$

Again, we have used the independence of X and Y to split the joint distribution $f_{X,Y}(x, y) = f_X(x)f_Y(y)$. Now, note that

$$f(t) = \begin{cases} 1, & \text{if } t \in (0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

This means that our integral will vanish whenever $t \leq 0$ or $t \geq 1$, so we can place our bounds as 0 and 1. Within this range, $f(t) = 1$, so we simplify

$$f_Z(z) = \int_0^1 f(z - t)\, dt.$$

If $z \le 0$, note that $z - t < 0$ where $t \in (0,1)$ so $f(z-t)$ vanishes, giving us zero. Again, if $z \ge 2$, we have $z - t > 2 - t > 1$ when $t \in (0,1)$, so again $f(z-t)$ vanishes giving us zero. When $1 \le z < 2$, we can only integrate where $0 < z - t < 1$, i.e. $z - 1 < t < 1$. When $0 < z < 1$, we integrate only where $0 < z - t < 1$, i.e. $0 < t < z$. Putting this together, and using

$$\int_0^z dt = z, \qquad \int_{z-1}^1 dt = 2 - z,$$

we have

$$f_Z(z) = \begin{cases} 0, & \text{if } z \le 0, \\ z, & \text{if } 0 < z < 1, \\ 2 - z, & \text{if } 1 \le z < 2, \\ 0, & \text{if } z \ge 2. \end{cases}$$

Again, the graph of such a function is a triangle with vertices at the origin $(0,0)$, $(1,1)$ and $(2,0)$. The area of such a triangle is

$$\int_{\mathbb{R}} f_Z(z)\, dz = \frac{1}{2} \cdot 2 \cdot 1 = 1.$$

Exercise 3 Let a random variable $X \sim \text{Poisson}(\lambda)$.

(a) Find the conditional expectation of X given that X is an even number.

(b) Find the variance of the conditional expectation of X given that X is even.

Solution Note that X has the probability mass function

$$p(x) = \frac{\lambda^n}{n!} e^{-\lambda}.$$

(a) Conditioning on X being even,

$$\mathrm{E}[X | X \text{ is even}] = \sum_{n=0}^{\infty} n \cdot \frac{P(X = n, n \text{ is even})}{P(X \text{ is even})}.$$

Now,

$$P(X \text{ is even}) = \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} e^{-\lambda}.$$

Use the facts that

$$e^{\lambda} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}, \qquad e^{-\lambda} = \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^n}{n!},$$

add to make the odd indexed terms disappear and the even terms double, and conclude

$$P(X \text{ is even}) = \frac{1}{2}(e^{\lambda} + e^{-\lambda}) e^{-\lambda} = \frac{1}{2}(1 + e^{-2\lambda}) = e^{-\lambda} \cosh \lambda.$$

Thus, we have

$$\mathrm{E}[X | X \text{ is even}] = \frac{2}{1 + e^{-2\lambda}} \sum_{n=0}^{\infty} (2n) \cdot \frac{\lambda^{2n}}{(2n)!} e^{-\lambda}.$$

The sum can be written as

$$\lambda e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^{2n-1}}{(2n-1)!}.$$

Using the same process as before, we conclude that the sum inside is

$$\frac{1}{2}(e^{\lambda} - e^{-\lambda}) = \sinh \lambda,$$

so

$$\mathrm{E}[X | X \text{ is even}] = \frac{2}{1 + e^{-2\lambda}} \lambda e^{-\lambda} \cdot \frac{1}{2}(e^{\lambda} - e^{-\lambda}) = \left(\frac{1 - e^{-2\lambda}}{1 + e^{-2\lambda}} \right) \cdot \lambda.$$

If desired, this can be further simplified as

$$\mathrm{E}[X | X \text{ is even}] = \lambda \tanh \lambda.$$

(b) Note that $\mathrm{E}[X|X \text{ is even}]$ is simply a real number, so $\mathrm{Var}[\mathrm{E}[X|X \text{ is even}]]$ is trivially zero.

For completeness, calculate

$$P(X \text{ is odd}) = 1 - P(X \text{ is even}) = \frac{1}{2}(1 - e^{-2\lambda}) = e^{-\lambda} \sinh \lambda,$$

and

$$\mathrm{E}[X|X \text{ is odd}] = \sum_{n=0}^{\infty} n \cdot \frac{P(X = n, n \text{ is odd})}{P(X \text{ is odd})} = \frac{2}{1 - e^{-2\lambda}} \sum_{n=0}^{\infty} (2n+1) \cdot \frac{\lambda^{2n+1}}{(2n+1)!} e^{-\lambda},$$

which we simplify as before to get

$$\frac{2}{1 - e^{-2\lambda}} \lambda e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} = \frac{2}{1 - e^{-2\lambda}} \cdot \lambda e^{-\lambda} \cdot \frac{1}{2}(1 + e^{-2\lambda}) = \left(\frac{1 + e^{-2\lambda}}{1 - e^{-2\lambda}} \right) \lambda.$$

Again, we may write

$$\mathrm{E}[X|X \text{ is odd}] = \lambda \coth \lambda.$$

Define the random variable

$$Y = \begin{cases} 1, & \text{if } X \text{ is even}, \\ 0, & \text{if } X \text{ is odd}. \end{cases}, \qquad p_Y(y) = \begin{cases} e^{-\lambda} \cosh \lambda, & \text{if } y = 1, \\ e^{-\lambda} \sinh \lambda, & \text{if } y = 0. \end{cases}$$

Thus,

$$\mathrm{E}[X|Y] = \begin{cases} \lambda \tanh \lambda, & \text{if } Y = 1, \\ \lambda \coth \lambda, & \text{if } Y = 0. \end{cases}$$

Thus,

$$\begin{aligned} \mathrm{E}[\mathrm{E}[X|Y]] &= \lambda \coth \lambda \cdot P(Y = 0) + \lambda \tanh \lambda \cdot P(Y = 1) \\ &= \lambda e^{-\lambda} \left(\coth \lambda \sinh \lambda + \tanh \lambda \cosh \lambda \right) \\ &= \lambda e^{-\lambda} \left(\cosh \lambda + \sinh \lambda \right) = \lambda, \end{aligned}$$

which we note is simply $\lambda = \mathrm{E}[X] = \mathrm{E}[\mathrm{E}[X|Y]]$, as demanded by the law of total expectation. Also,

$$\begin{aligned} \mathrm{E}[\mathrm{E}[X|Y]^2] &= \lambda^2 \coth^2 \lambda \cdot P(Y = 0) + \lambda^2 \tanh^2 \lambda \cdot P(Y = 1) \\ &= \lambda^2 e^{-\lambda} \left(\coth^2 \lambda \sinh \lambda + \tanh^2 \lambda \cosh \lambda \right) \\ &= \lambda^2 e^{-\lambda} \left(\frac{\cosh^2 \lambda}{\sinh \lambda} + \frac{\sinh^2 \lambda}{\cosh \lambda} \right) \\ &= \lambda^2 e^{-\lambda} \cdot \frac{\cosh^3 \lambda + \sinh^3 \lambda}{\cosh \lambda \sinh \lambda}. \end{aligned}$$

Now,

$$\cosh^3 \lambda = \frac{1}{8}(e^{3\lambda} + 3e^{\lambda} + 3e^{-\lambda} + e^{-3\lambda}), \qquad \sinh^3 \lambda = \frac{1}{8}(e^{3\lambda} - 3e^{\lambda} + 3e^{-\lambda} - e^{-3\lambda}),$$

so

$$\cosh^3 \lambda + \sinh^3 \lambda = \frac{1}{4}(e^{3\lambda} + 3e^{-\lambda}).$$

Also,

$$\cosh \lambda \sinh \lambda = \frac{1}{4}(e^{2\lambda} - e^{-2\lambda}).$$

Thus,

$$\mathrm{E}[\mathrm{E}[X|Y]^2] = \lambda^2 \frac{e^{2\lambda} + 3e^{-2\lambda}}{e^{2\lambda} - e^{-2\lambda}} = \lambda^2 \left(1 + \frac{4e^{-2\lambda}}{e^{2\lambda} - e^{-2\lambda}} \right).$$

Putting these together,

$$\mathrm{Var}[\mathrm{E}[X|Y]] = \mathrm{E}[\mathrm{E}[X|Y]^2] - \mathrm{E}[X]^2 = \frac{4\lambda^2 e^{-2\lambda}}{e^{2\lambda} - e^{-2\lambda}}.$$