MA 2202 : Probability I

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Exercise 1 Observe that for a unit normal distribution, $P(X > 2) \approx 0.02275$. In this in accordance with the 3σ rule of thumb?

Solution The 3σ rule of thumb claims that

$$P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx 0.9545.$$

In the complementary region,

$$P(|X - \mu| > 2\sigma) \approx 1 - 0.9545 \approx 0.0455.$$

In our particular case, we have $\mu = 0$ and $\sigma^2 = 1$, $P(X > 2) \approx 0.02275$. Symmetry of the normal distribution about the origin means that $P(X < -2) \approx 0.02275$ as well, so

$$P(|X - 0| \geq 2) = P(X < -2) + P(X > 2) \approx 2 \times 0.02275 \approx 0.0455.$$

This is in perfect accordance with our rule of thumb.

Exercise 2 Find the moment generating function for the exponential random variable with parameter λ.

Solution We have a random variable X such that

$$f_X(x) = \lambda e^{-\lambda x} u(x),$$

where $u(x)$ is the Heaviside step function and $\lambda > 0$. Now,

$$M_X(t) = E[e^{tX}] = \int_0^\infty \lambda e^{tx} e^{-\lambda x}\, dx.$$

This converges only when $t < \lambda$, so

$$M_X(t) = \frac{\lambda}{t - \lambda} e^{-(\lambda - t)x} \Big|_0^\infty = \frac{\lambda}{\lambda - t}.$$

NOTE: In the domain $t \in (0, \lambda)$, we can write

$$M_X(t) = \frac{1}{1 - t/\lambda} = 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \cdots + \frac{t^n}{\lambda^n} + \ldots.$$

This directly gives

$$E[X^n] = \frac{n!}{\lambda^n}.$$

Putting $\lambda = 1$, we have shown that

$$E[X^n] = \int_0^\infty x^n e^{-x}\, dx = \Gamma(n + 1) = n!.$$

Exercise 3 Let $X \sim \text{Normal}(0, 1)$ and let Φ denote the cumulative distribution function of X.

(a) Show that for all $a > 0$,

$$\Phi(a) \geq \frac{a^2}{1 + a^2}.$$

(b) For $r \in \mathbb{R}$, compute $E[e^{rX}]$.

Solution

(a) Note that

$$\Phi(a) = P(X \le a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-t^2/2}\, dt.$$

Since $1 - \Phi(a) = P(X \ge a)$, we instead prove the equivalent statement that

$$P(X \ge a) = \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} e^{-t^2/2}\, dt \le 1 - \frac{a^2}{1+a^2} = \frac{1}{1+a^2}.$$

When $a \le 1$, this is trivial. Note that our integral is symmetric about $x = 0$, so the integral over $(0, \infty)$ is just half of the integral over $\mathbb{R}$, which must be 1. Thus,

$$\frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} e^{-t^2/2}\, dt \le \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t^2/2}\, dt = \frac{1}{2} \le \frac{1}{1+a^2}.$$

The last inequality holds when $a \le 1$, since $2 \ge 1 + a^2$.

Now consider $a > 1$. When we integrate, we have $a \le t$ for the dummy variable t. Thus,

$$\frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} a e^{-t^2/2}\, dt \le \frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} t e^{-t^2/2}\, dt = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \Big|_{a}^{\infty} = \frac{1}{\sqrt{2\pi}} e^{-a^2/2}.$$

This gives

$$\frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} e^{-t^2/2}\, dt \le \frac{e^{-a^2/2}}{\sqrt{2\pi} a}.$$

When $a > 1$, we have $\sqrt{2\pi} a > 2$. Also,

$$e^{a^2/2} > 1 + \frac{1}{2} a^2,$$

so

$$\sqrt{2\pi} a e^{a^2/2} > 2 e^{a^2/2} > 2 + a^2 > 1 + a^2.$$

Putting this together,

$$\frac{1}{\sqrt{2\pi}} \int_{a}^{\infty} e^{-t^2/2}\, dt \le \frac{1}{\sqrt{2\pi} a e^{a^2/2}} < \frac{1}{1+a^2}.$$

This proves the desired inequality for all $a > 0$.

(b) We want to compute

$$E[e^{rX}] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{rt} e^{-t^2/2}\, dt.$$

Completing the square,

$$\frac{1}{2} t^2 - rt = \frac{1}{2}(t-r)^2 - \frac{1}{2} r^2,$$

so

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(-rt + t^2/2)}\, dt = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{r^2/2} e^{-(t-r)^2/2}\, dt.$$

We take the factor of $e^{r^2/2}$ outside and shift the variable of integration as $t \mapsto t + r$. The result is simply

$$E[e^{rX}] = e^{r^2/2} \cdot \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-t^2/2}\, dt = e^{r^2/2}.$$