# MA 2202 : Probability I

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**Exercise 1** If the probability distribution of a non-negative discrete random variable $X$ is memoryless, show that $X \sim \text{Geometric}(p)$ for some $p \in [0,1]$.

**Solution** Set $P(X = 1) = p \in [0,1]$ and $q = 1 - p$. Now, the memorylessness of $X$ means that for all $m, n \in \mathbb{N}_0$,

$$P(X > m + n \mid X > n) = P(X > m), \qquad P(X > m + n) = P(X > m)\, P(X > n).$$

We claim that

$$P(X > n) = q^n$$

for all $n \in \mathbb{N}$. First, note that

$$1 = P(X > 0 \mid X > 0) = P(X > 0), \qquad P(X = 0) = 1 - P(X > 0) = 0,$$

so

$$P(X > 1) = 1 - (P(X = 0) + P(X = 1)) = 1 - p = q^1,$$

which establishes our base case. Now, if $P(X > k) = q^k$ for some $k > 1$, then

$$P(X > k + 1) = P(X > k)\, P(X > 1) = q^k \cdot q = q^{k+1},$$

which establishes $P(X > n) = q^n$ by induction. Now,

$$P(X = n) = P(X > n - 1) - P(X > n) = q^{n-1} - q^n = (1 - q)q^{n-1} = pq^{n-1},$$

which means that $X \sim \text{Geometric}(p)$ as desired.

**Exercise 2** If the probability distribution of a non-negative continuous random variable $X$ is memoryless, show that $X \sim \text{Exponential}(\lambda)$ for some $\lambda > 0$.

**Solution** Choose $\lambda \geq 0$ such that $P(X > 1) = e^{-\lambda} < 1$. The memorylessness of $X$ means that for all $s, t > 0$,

$$P(X > s + t \mid X > t) = P(X > s), \qquad P(X > s + t) = P(X > s)\, P(X > t).$$

We claim that the cumulative distribution function of $X$ must be of the form

$$P(X > s) = e^{-\lambda s}.$$

First, note that $P(X \geq 0) = P(X > 0) = 1$ from the non-negativity of $X$. Also, $P(X > 1) = e^{-\lambda}$. To show this holds for all natural numbers, suppose that $P(X > k) = e^{-\lambda k}$ for some $k \in \mathbb{N}$. Then,

$$P(X > k + 1) = P(X > k)\, P(X > 1) = e^{-\lambda k} \cdot e^{-\lambda} = e^{-\lambda(k+1)}$$

which establishes $P(X > n) = e^{-\lambda n}$ for all $n \in \mathbb{N}_0$. Now, for any real number $x$ and $m \in \mathbb{N}$, note that

$$P(X > mx) = P(X > x) \cdot P(X > (m-1)x) = \cdots = [P(X > x)]^m.$$

This means that for any rational $p/q$ where $p, q \in \mathbb{N}$, we have

$$P(X > p) = [P(X > p/q)]^q, \qquad P(X > p/q) = [P(X > p)]^{1/q} = \left[e^{-\lambda p}\right]^{1/q} = e^{-\lambda p/q},$$

so $P(X > r) = e^{-\lambda r}$ for all positive rationals $r \in \mathbb{Q}_+$. Now, the rationals are dense in the reals and the cumulative distribution function of a continuous random variable is continuous (hence so is $P(X > x) = 1 - P(X \leq x)$). This means that we can find a sequence of rationals $r_n \to x$, so

$$P(X > x) = \lim_{n\to\infty} P(X > r_n) = \lim_{n\to\infty} e^{-\lambda r_n} = e^{-\lambda x}.$$

Thus, $P(X > x) = e^{-\lambda x}$ for all non-negative real numbers $x \in \mathbb{R}_+$. Furthermore, note that

$$P(X \leq x) = 1 - e^{-\lambda x} = \int_0^x \lambda e^{-\lambda t}\, dt = \int_{-\infty}^{+\infty} \lambda e^{-\lambda t}\, u(t)\, dt,$$

where $u(t)$ is the step function which assumes the value 0 for all $x < 0$ and 1 for all $x \geq 0$. Thus, we have found a probability density function

$$f_X(x) = \lambda e^{-x}\, u(x),$$

which is precisely that of an exponential distribution. Hence, $X \sim \text{Exponential}(\lambda)$. Finally, we eliminate $\lambda = 0$ since that gives an identically zero probability density function.

**Exercise 3** Let $X$ be a random variable. Find $E[X]$ in each of the following cases.

(a) $X \sim \text{Geometric}(p)$.
(b) $X \sim \text{Poisson}(\lambda)$.
(c) $X \sim \text{Exponential}(\lambda)$.
(d) $X \sim \text{Pascal}(n, p)$.

**Solution** We state and prove the following lemma.

**Lemma.** *If $X$ is a non-negative discrete random variable with finite expectation, then*

$$E[X] = \sum_{n=0}^{\infty} P(X > n).$$

*Proof.* Note that

$$P(X > n-1) = P(X = n) + P(X > n).$$

Thus,

$$E[X] = \sum_{n=1}^{\infty} n\, P(X = n) = \sum_{n=1}^{\infty} n\, [P(X > n-1) - P(X > n)] = \sum_{n=1}^{\infty} n\, P(X > n-1) - \sum_{n=1}^{\infty} n\, P(X > n).$$

Reshuffling indices,

$$E[X] = \sum_{n=0}^{\infty} (n+1)\, P(X > n) - \sum_{n=1}^{\infty} n\, P(X > n) = P(X > 0) + \sum_{n=1}^{\infty} (n + 1 - n)\, P(X > n),$$

which gives

$$E[X] = \sum_{n=0}^{\infty} P(X > n).$$ □

(a) We have the probability mass function

$$P(X = n) = pq^{n-1},$$

where $q = 1 - p$ and $n = 1, 2, \ldots$. Now,

$$P(X > n) = \sum_{k=n+1}^{\infty} pq^{k-1} = pq^n \cdot \frac{1}{1-p} = q^n.$$

Thus,

$$E[X] = \sum_{n=0}^{\infty} P(X > n) = \sum_{n=0}^{\infty} q^n = \frac{1}{1-q} = \frac{1}{p}.$$

Note that $p > 0$.

(b) We have the probability mass function

$$P(X = n) = \frac{\lambda^n}{n!} e^{-\lambda}.$$

Now,

$$E[X] = \sum_{n=1}^{\infty} n \cdot \frac{\lambda^n}{n!} e^{-\lambda} = \lambda e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} = \lambda e^{-\lambda} e^{\lambda} = \lambda.$$

(c) We have the probability density function

$$f_X(x) = \lambda e^{-\lambda x}.$$

Thus,

$$E[X] = \int_0^\infty x\, f_X(x)\, dx = \int_0^\infty \lambda x e^{-\lambda x}\, dx = -xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x}\, dx = \frac{1}{\lambda} e^{-\lambda x}\Big|_0^\infty = \frac{1}{\lambda}.$$

(d) We have the probability mass function

$$P(X = k) = \binom{k-1}{n-1} p^n q^{k-n}$$

where $q = 1 - p$ and $k \geq n$. Now, note that $X \sim \text{Pascal}(n, p)$ represents the number of Bernoulli trials required for getting $n$ successes. If $X_i \sim \text{Geometric}(p)$, then each $X_i$ denotes the number of Bernoulli trials until 1 success. Thus, we can write

$$X = X_1 + X_2 + \cdots + X_n,$$

since the total number of trials must add up. The set of $X_i$ contains identical and independent random variables. Using linearity of expectation (shown in the next problem),

$$E[X] = \sum_{k=1}^{n} E[X_k] = \frac{n}{p}.$$

**Exercise 4**

(a) Let $X_1, X_2, \ldots, X_n$ be random variables which are all defined on the same sample space $\Omega$, each of which has finite expectation. Show that

$$E[a_1 X_1 + \cdots + a_n X_n] = a_1 E[X_1] + \cdots + a_n E[X_n]$$

for all $a_1, a_2, \ldots, a_n \in \mathbb{R}$.

(b) Deduce the expectation value of a Binomial random variable from the expectation of a Bernoulli random variable.

**Solution**

(a) First, let $X_i$ be discrete random variables. Now, for discrete variables $X$ and $Y$ and $a \in \mathbb{R}$,

$$E[aX] = \sum_x (ax)\, P(aX = ax) = a \sum_x x\, P(X = x) = aE[X],$$

and

$$\begin{aligned} E[X + Y] &= \sum_{x,y} (x + y)\, P(X = x, Y = y) \\ &= \sum_{xy} x\, P(X = x, Y = y) + \sum_{xy} y\, P(X = x, Y = y) \\ &= \sum_x x\, P(X = x) + \sum_y y\, P(Y = y) \\ &= E[X] + E[Y]. \end{aligned}$$

We have used the fact that

$$\sum_{xy} x\, P(X=x, Y=y) = \sum_x x \sum_y P(X=x, Y=y) = \sum_x x P(X=x).$$

Using these two rules finitely many times, we have

$$E[a_1 X_1 + a_2 X_2 + \cdots + a_n X_n] = a_1 E[X_1] + E[a_2 X_2 + \cdots + a_n X_n] = \cdots = a_1 E[X_1] + \cdots + a_n E[X_n].$$

Now, let $X_i$ be continuous random variables. Again,

$$E[aX] = \int_{\mathbb{R}} ax\, f_X(x)\, dx = aE[X].$$

Also,

$$\begin{aligned} E[X+Y] &= \iint_{\mathbb{R}^2} (x+y)\, f_{X,Y}(x,y)\, dx\, dy \\ &= \iint_{\mathbb{R}^2} x\, f_{X,Y}(x,y)\, dx\, dy + \iint_{\mathbb{R}^2} y\, f_{X,Y}(x,y)\, dy\, dx \\ &= \int_{\mathbb{R}} x\, f_X(x)\, dx + \int_{\mathbb{R}} y\, f_Y(y)\, dy \\ &= E[X] + E[Y]. \end{aligned}$$

Here, we have used

$$\int_{\mathbb{R}} f_{X,Y}(x,y)\, dy = f_X(x).$$

Like before, we use these two rules finitely many times to get

$$E[a_1 X_1 + a_2 X_2 + \cdots + a_n X_n] = \cdots = a_1 E[X_1] + \cdots + a_n E[X_n].$$

(b) Note that if $X \sim \text{Binomial}(n,p)$, we can write it as the sum of $X_i \sim \text{Bernoulli}(p)$ as

$$X = X_1 + \cdots + X_n.$$

The expectation of a Bernoulli random variable is

$$E[X_i] = 0 \cdot (1-p) + 1 \cdot p = p.$$

Thus, linearity of expectation gives the expectation of the Binomial random variable as

$$E[X] = \sum_{k=1}^{n} E[X_k] = np.$$