MA 2202 : Probability I

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Exercise 1 Show that if you keep on tossing a fair coin, eventually you would get a head for sure.

Solution Let X be the random variable denoting the number of tosses until a head is first obtained. Note that if $X = n$, this means that the first $n-1$ tosses must have been tails and the n^{th} must have been a head. Multiplying, we have

$$P(X = n) = \frac{1}{2^n}.$$

By 'eventually get a head for sure', we mean that the probability of having obtained a head at some point of time becomes closer and closer to certain as the total number of tosses increases, i.e. as $n \to \infty$, the probability that we never encountered a head becomes closer and closer to 0, so we want $P(X = n) \to 0$. This is indeed the case because $1/2^n \to 0$.

Remark. This does not mean that if one person starts tossing a coin repeatedly, they cannot get a sequence of all tails, i.e. the sequence $TTT\ldots$ ought to be a legitimate outcome. It's just that this particular outcome is associated with probability 0. There is no other value we can assign, since the probability of this outcome can only be the limit $P(X = n \to \infty)$, and this particular sequence is the limit of the sequence of outcomes $T^{n-1}H$.

Exercise 2 Let $\Omega = \{1, 2, 3, 4, 5\}$ and $\mathcal{E} = \{\emptyset, \Omega, \{1\}, \{2, 3, 4, 5\}\}$. Define

$$X : \Omega \to \mathbb{R}, \qquad X(\omega) = \omega + 1$$

for all $\omega \in \Omega$. Is X a random variable?

Solution No. Recall that for X to be a random variable, we must have $X^{-1}((r, \infty)) \in \mathcal{E}$ for all $r \in \mathbb{R}$. On the other hand, $X^{-1}((3, \infty)) = \{3, 4, 5\} \notin \mathcal{E}$.

Exercise 3 Let $(\Omega, \mathcal{E}, P)$ be a probability space and let $A_1, A_2, \ldots, \in \mathcal{E}$ be pairwise mutually exclusive. Let $A = \cup_{n=1}^{\infty} A_n$ and let $B \in \mathcal{E}$ with $P(B) \neq 0$. Show that

$$P(A \mid B) = \sum_{n=1}^{\infty} P(A_n \mid B).$$

Solution Note that since $\{A_n\}$ are pairwise exclusive, so are $\{A_n \cap B\}$ since

$$(A_i \cap B) \cap (A_j \cap B) = (A_i \cap A_j) \cap B = \emptyset.$$

Therefore, the probability of the union of these countably many disjoint events is simply the sum of their probabilities, so

$$P\left(\bigcup_{n=1}^{\infty} A_n \cap B\right) = \sum_{n=1}^{\infty} P(A_n \cap B). \qquad (\star)$$

Note that[1]

$$\bigcup_{n=1}^{\infty} A_n \cap B = B \cap \bigcup_{n=1}^{\infty} A_n = B \cap A,$$

[1] To see this formally, note that the countably infinite unions are well defined since the event space is closed under such unions. First, pick $x \in \cup_{n=1}^{\infty} A_n \cap B$, which means that $x \in A_n \cap B$ for some $n \in \mathbb{N}$. This means that $x \in A_n$ and $x \in B$, so $x \in \cup_{n=1}^{\infty} A_n$ whence $x \in B \cap \cup_{n=1}^{\infty} A_n$.

For the reverse direction, pick $x \in B \cap \cup_{n=1}^{\infty} A_n$, which means that $x \in B$ and $x \in \cup_{n=1}^{\infty} A_n$. Thus, $x \in A_n$ for some $n \in \mathbb{N}$, so $x \in B \cap A_n$ whence $x \in \cup_{n=1}^{\infty} A_n \cap B$.

and

$$P(X \mid B) = \frac{P(X \cap B)}{P(B)}.$$

Thus, dividing ($\star$) by $P(B)$, we obtain the desired result.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \sum_{n=1}^{\infty} \frac{P(A_n \cap B)}{P(B)} = \sum_{n=1}^{\infty} P(A_n \mid B).$$

Exercise 4 The probability of a family having exactly k children is p_k, where $\sum_{k=1}^{\infty} p_k = 1$. We assume the gender of each child is either male, female or the third gender with equal probability.

(a) What is the probability that the family has only girls?

(b) Now, if it is known that the family has no girls, what is the probability that the family has only one child?

Solution

(a) The probability that a family having exactly k children has only girls is simply $1/3^k$, since the probability of a child being a girl is $1/3$. Now, for a randomly chosen family, we can write

$$P(\text{all girls}) = \sum_{n=1}^{\infty} P(\text{all girls} \mid k \text{ children})\, P(k \text{ children}) = \sum_{k=1}^{\infty} \frac{p_k}{3^k}.$$

(b) The probability that the family has only 1 child is p_1, and the probability that a family with k children has no girls is $(2/3)^k$, since each child must be either a boy or third gender. Thus, the probability of a family having no girls is set to

$$P_{ng} = P(\text{no girls}) = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^k p_k.$$

Thus, Bayes' Theorem gives

$$P(1 \text{ child} \mid \text{no girls}) = \frac{P(1 \text{ child})}{P(\text{no girls})} P(\text{no girls} \mid 1 \text{ child}) = \frac{p_1}{P_{ng}} \cdot \frac{2}{3}.$$

Exercise 5 Suppose you are a contestant in a game, where you have the potential to win an expensive car, by making the right decision and win. There are 3 closed doors in front of you and the host explains to you the rules of the game. Two of the three doors have goats behind them and the third one has the valuable car behind it. You have to choose one of the three doors to start the game. Then, from the two other doors, the host will reveal one of that which has a goat behind it. Then you will be asked, whether you want to open the door that you chose, or switch to the other. If you choose wisely, the door that has the car behind it, you will win and get your prize. So, should you switch the door or would you wish to keep your choice of the door unchanged? Justify your answer.

Solution Always switch. The probability of picking the car initially is $1/3$. The host always reveals a goat, so the probability that your initially chosen door still has the car behind it is given as

$$P(\text{car} \mid \text{goat reveal}) = \frac{P(\text{car})}{P(\text{goat reveal})} P(\text{goat reveal} \mid \text{car}) = \frac{1/3}{1} \cdot 1 = \frac{1}{3}.$$

Thus, the probability that the remaining door has the car behind it must be $1 - 1/3 = 2/3$, since there are no other possible outcomes. This gives better odds of winning on switching[2].

[2]This is the well known Monty Hall problem.