MA 2202 : Probability I

Satvik Saha, 19MS154, Group D January 28, 2021

Exercise 1 Compute the probability of obtaining a total of 10 points when two unbiased dice are rolled.

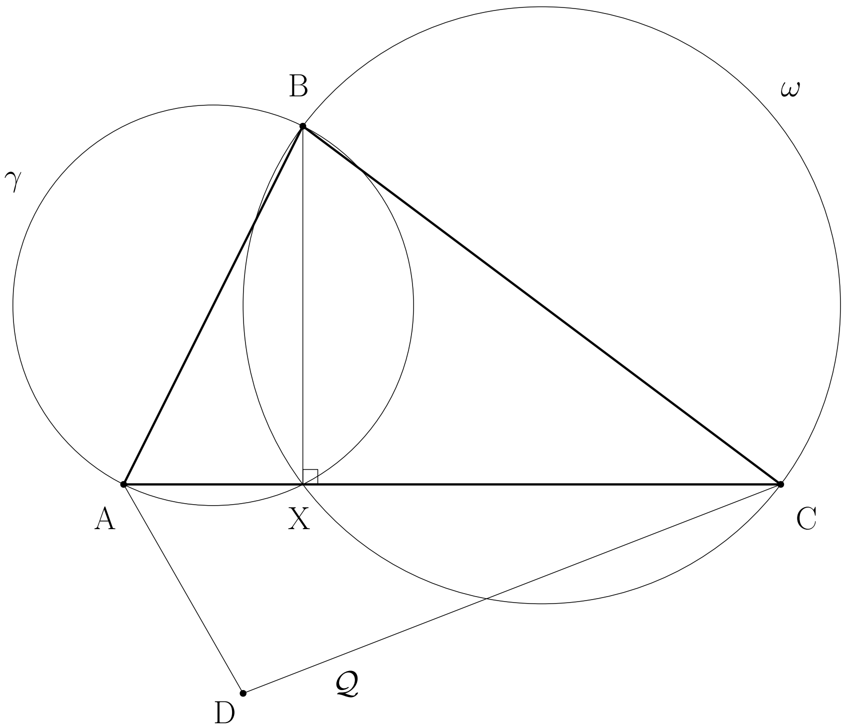
Solution The sample space of the possible ordered rolls is given by

$$\Omega = \{(m, n) \colon m, n \in \mathbb{N}, 1 \leq m, n \leq 6\}.$$

Since each of the individual rolls of a die is equally likely and the die are independent of one another, we may write that each of the outcomes (m, n) is equally likely. There are $6 \times 6 = 36$ such tuples in Ω, hence the probability of each of them is $1/36$. Note that we may set $\mathcal{E} = \mathcal{P}(\Omega)$. Now, we want $m + n = 10$, which is satisfied precisely by the tuples $(4, 6)$ $(5, 5)$ and $(6, 4)$. Hence, the desired probability is simply $3/36 = 1/12$.

Exercise 2 Let x be a point inside a convex quadrilateral $\mathcal{Q}$. Find the probability that x is neither on the boundary nor inside any of the circles drawn with the sides of the quadrilateral Q as their diameters.

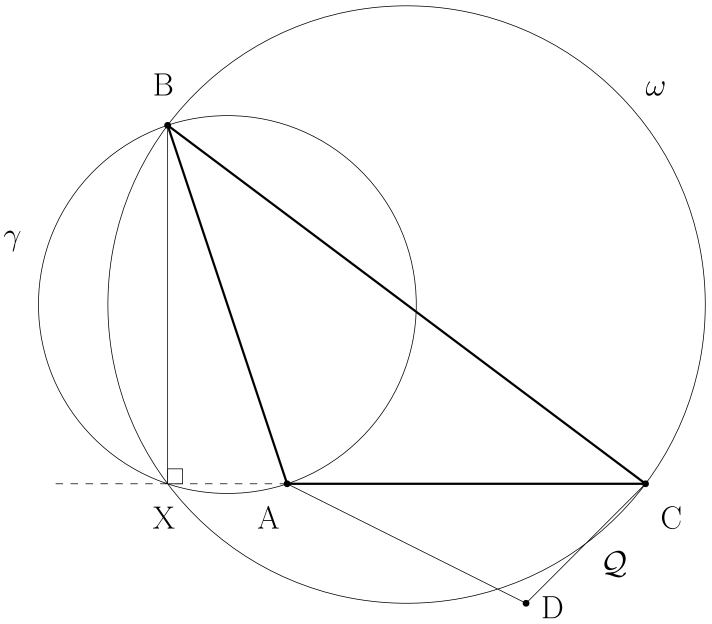
Solution We claim that the required probability is 0, i.e. there are no points within $\mathcal{Q}$ which lie outside the constructed circles. To show this, pick an arbitrary point $P \in \mathcal{Q}$. Draw a diagonal AC of $\mathcal{Q}$; the convexity of $\mathcal{Q}$ guarantees that the segment AC lies entirely within $\mathcal{Q}$. Choose the triangle ΔABC on the side of AC where P lies, or choose the vertex B arbitrarily if P lies on the diagonal AC; thus, P lies within (or on) ΔABC.



Now, construct a circle γ with AB as its diameter, and let it intersect the line AC at the point X. Note that the triangle ΔABX is inscribed within γ, and the side AB is a diameter of γ. This means that ΔABX is right angled, with $\angle AXB = \pi/2$. Thus, BX is the perpendicular through B dropped on AC. Mirroring the same argument on the other side with a circle ω with diameter CB, we see that ω must cut AC at the same point X, since there is only one perpendicular that can be dropped through B

onto AC. Thus, the triangles ΔABX and ΔCBX both lie within γ and ω (including their boundaries) so the triangle ΔABC lies within the union of γ and ω. This means that P must also lie within one of the circles. Since P and $\mathcal{Q}$ were arbitrary, there are no possible scenarios where P lies outside the constructed circles, hence the desired probability is 0.

Note that the given argument holds even when BX lies outside the triangle ΔABC – it so happens that ΔABC is completely within one of the triangles ΔABX or ΔCBX.



It also holds for any concave quadrilateral $\mathcal{Q}'$, since we can always draw a diagonal AC lying outside $\mathcal{Q}'$; now, the entire quadrilateral is contained within ΔABC where B is the furthest vertex from AC, and we argue as before to show that ΔABC lies within the circles constructed with AB and CB as diameters.

Exercise 3 Compute the probability of getting no four consecutive heads or no four consecutive tails when a fair coin is tossed ten times.

Solution Let G_n denote the number of times a string of length n consisting only of the characters H and T which do not contain either of the substrings $HHHH$ or $TTTT$. Note that for $n = 1, 2, 3$, all possible strings of which there are 2^n satisfy the criterion, so $G_1 = 2$, $G_2 = 4$ and $G_3 = 8$.

Now, let $G_{n,i}$ denote that number of valid strings which end in exactly i identical characters. Note that

$$G_n = G_{n,1} + G_{n,2} + G_{n,3},$$

since $i = 4$ gives an invalid string. By adding another identical character to the end of a valid string (H if the last i characters were H, T otherwise), we obtain $G_{n+1,i+1} = G_{n,i}$. By adding the complementary character to the end of a valid string (T if the last i characters were H, and vice versa), we obtain another valid string of $n+1$ characters with $i = 1$, so $G_{n+1,1} = G_n$. This gives a complete recursive solution : $G_{n+3,1} = G_{n+2}$, $G_{n+3,2} = G_{n+2,1} = G_{n+1}$ and $G_{n+3,3} = G_{n+1,1} = G_n$, so

$$G_{n+3} = G_{n+3,1} + G_{n+3,2} + G_{n+3,3} = G_{n+2} + G_{n+1} + G_n.$$

We calculate $G_4 = 2+4+8 = 14$, $G_5 = 4+8+14 = 26$, $G_6 = 8+14+26 = 48$, $G_7 = 14+26+48 = 88$, $G_8 = 26 + 48 + 88 = 162$, $G_9 = 48 + 88 + 162 = 298$, $G_{10} = 88 + 162 + 298 = 548$. Thus, there are 548 valid strings out of $2^{10} = 1024$ strings of length 10, all of which are equally likely to occur in a coin toss experiment. This means that the desired probability is $548/1024 = 137/256 \approx 53.5\%$.

NOTE: The sequence G_n is simply twice the Tribonacci sequence $0, 0, 1, 1, 2, 4, 7, 13, 24, \ldots$, with a shift. As shown here, we can give the closed form expression

$$\frac{1}{2}G_n = \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)},$$

where α, β, γ are the roots of the polynomial $x^3 - x^2 - x - 1$. We supply

$$\begin{aligned} \alpha &\approx +1.8393, \\ \beta &\approx -0.4196 + 0.6063i, \\ \gamma &\approx -0.4196 - 0.6063i. \end{aligned}$$

Some code used for these calculations can be found here.

The ratio G_{n+1}/G_n converges as $n \to \infty$ to the constant α. This can be seen by the fact that $|\beta| < 1$ and $|\gamma| < 1$, so the growth of G_n is dominated by α^n. On the other hand, the number of strings of length n grows as 2^n. Thus, the probability of not getting a repeated substring of length 4 can be made as small as we want, i.e. the probability of obtaining a stretch of 4 identical tosses can be made as close to 1 by increasing n. Indeed, by $n = 30$, the probability of not getting a repeat is around 10%, and by $n = 57$, the probability is around 1%.

The corresponding limiting ratio for a repeat of k tosses always satisfies $2(1 - 2^{-k}) < \alpha_k < 2$, so the same argument above applies to show that we will always obtain a repeat of k tosses more often than not, for a sufficiently high number of total tosses n.

Exercise 4 Let $(\Omega, \mathcal{E}, P)$ be a probability space and let $A_1, A_2, \ldots, A_n \in \mathcal{E}$ with $P(A_1 \cap \cdots \cap A_n) \neq 0$. Show that

$$P(A_1 \cap \cdots \cap A_n) = P(A_1)\, P(A_2 \,|\, A_1)\, P(A_3, | A_2 \cap A_1) \cdots P(A_n \,|\, A_{n-1} \cap \cdots \cap A_1).$$

Solution We prove the given statement by induction. The base case of $n = 2$ demands

$$P(A_1 \cap A_2) = P(A_1)\, P(A_2 \,|\, A_1),$$

which is true by definition. Note that since $P(\cap_{i=1}^{n} A_i) \neq 0$, we must have $P(\cap_{i=1}^{m \leq n} A_i) \neq 0$ because $P(B) \leq P(A)$ whenever $B \subseteq A$.

Suppose that the statement holds for some $n \geq 2$. Write $\cap_{i=1}^{n} A = \mathcal{A}$, hence

$$P(\mathcal{A} \cap A_{n+1}) = P(\mathcal{A})\, P(A_{n+1} \,|\, \mathcal{A}).$$

Expanding $P(\mathcal{A})$ using the induction hypothesis gives the desired result.

$$P\left(\bigcap_{i=1}^{n+1} A_i\right) = P(A_1)\, P(A_2 \,|\, A_1) \cdots P(A_n \,|\, A_{n-1} \cap \cdots \cap A_1)\, P(A_{n+1} \,|\, A_n \cap \cdots \cap A_1).$$