MA2201: Analysis II

Integration

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Definition 3.1 (Partition)**.** A partition P of an interval $[a, b]$ is a finite sequence of numbers

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

The norm of a partition is defined as

$$\|P\| = \max |x_{j+1} - x_j|.$$

Definition 3.2 (Tagged partition)**.** A tagged partition $\dot{P}(x_j, \xi_j)$ is a partition P together with a set of numbers ξ_j such that $\xi_j \in [x_j, x_{j+1}]$.

Definition 3.3 (Riemann sum)**.** The Riemann sum of a function f on an interval $[a, b]$ with respect to a tagged partition $\dot{P}$ is defined as

$$S(f, \dot{P}) = \sum_{j=0}^{n-1} f(\xi_j)(x_{j+1} - x_j).$$

Definition 3.4 (Riemann integral)**.** A function f is called Riemann integrable on an interval $[a, b]$ if there is some $\ell \in \mathbb{R}$ where for every $\epsilon > 0$, there exists $\delta > 0$ such that all tagged partitions $\dot{P}$ of $[a, b]$ with $\|\dot{P}\| < \delta$ satisfy

$$|S(f, \dot{P}) - \ell| < \epsilon.$$

The number ℓ is the value of the Riemann integral,

$$\int_a^b f = \ell.$$

Theorem 3.1. *If a function is Riemann integrable on an interval, then the value of the integral is unique.*

Proof. Let f be Riemann integrable on $[a,b]$, with integral values ℓ and ℓ'. Then, for every $\epsilon > 0$, we find $\delta > 0$ such that for all tagged partitions $\dot{P}$ with $\|\dot{P}\| < \delta$,

$$|S(f,\dot{P}) - \ell| < \frac{\epsilon}{2}, \qquad |S(f,\dot{P}) - \ell'| < \frac{\epsilon}{2}.$$

Note that such a partition $\dot{P}$ always exists. Thus,

$$|\ell - \ell'| \leq |S(f,\dot{P}) - \ell| + |S(f,\dot{P}) - \ell'| < \epsilon$$

for all $\epsilon > 0$, which forces $\ell = \ell'$. □

Theorem 3.2. *If f is Riemann integrable on $[a,b]$, then f is bounded on that interval. Furthermore, if $M > 0$ is such that $|f(x)| \leq M$ for all $x \in [a,b]$, then*

$$-M(b-a) \leq \int_a^b f \leq M(b-a).$$

Proof. Suppose not. Let the Riemann integral of f on $[a,b]$ be ℓ. For $\epsilon = 1$, we find $\delta > 0$ such that for all tagged partitions $\dot{P}$ of $[a,b]$ with $\|\dot{P}\| < \delta$, we have $|S(f,\dot{P}) - \ell| < 1$. This means that

$$S(f,\dot{P}) < |\ell| + 1.$$

Let $Q = \{x_0, \dots, x_n\}$ be such a partition. The unboundedness of f means that we can find a subinterval $[x_k, x_{k+1}]$ where f is unbounded. Now, choose tags ξ_j creating the tagged partition $\dot{Q}$. We choose the tag $\xi_k \in [x_k, x_{k+1}]$ such that

$$|f(\xi_k)(x_{k+1} - x_k)| > |\ell| + 1 + |\sum_{j \neq k} f(\xi_j)(x_{j+1} - x_j)|.$$

Now, observe that the triangle inequality demands

$$|S(f,\dot{Q})| \geq |f(\xi_k)(x_{k+1} - x_k)| - |\sum_{j \neq k} f(\xi_j)(x_{j+1} - x_j)| > |\ell| + 1,$$

which is a contradiction. Thus, f must be bounded on $[a,b]$.

Next, for any tagged partition $\dot{P}$ of $[a,b]$, we have

$$|S(f,\dot{P})| \leq \sum_{j=0}^{n-1} |f(\xi_j)|(x_{j+1} - x_j) \leq M(b-a).$$

Let the Riemann integral of f be ℓ. Thus, for all $\epsilon > 0$, we find $\delta > 0$ such that for all tagged partitions $\dot{P}$ with $\|\dot{P}\| < \delta$,

$$||S(f,\dot{P})| - |\ell|| \leq |S(f,\dot{P}) - \ell| < \epsilon.$$

This gives

$$|\ell| < |S(f,\dot{P})| + \epsilon \leq M(b-a) + \epsilon.$$

Since this holds for all $\epsilon > 0$, we may write

$$|\ell| \leq M(b-a).$$ □

Theorem 3.3. *If f is Riemann integrable on $[a, b]$, and $\dot{P}_n$ is any sequence of tagged partitions of $[a, b]$ such that $\|\dot{P}_n\| \to 0$, then*

$$\int_a^b f = \lim_{n\to\infty} S(f, \dot{P}_n).$$

Proof. Let $\epsilon > 0$. We find $\delta > 0$ such that for all tagged partitions $\dot{P}$ with $\|\dot{P}\| < \delta$, we have

$$|S(f, \dot{P}) - \int_a^b f| < \epsilon.$$

Now, since $\|\dot{P}_n\| \to 0$, we can choose $N \in \mathbb{N}$ such that for all $n \geq N$, $\|\dot{P}_n\| < \delta$. Thus, for all $n \geq N$,

$$|S(f, \dot{P}_n) - \int_a^b f| < \epsilon.$$

In other words,

$$\int_a^b f = \lim_{n\to\infty} S(f, \dot{P}_n).$$

□

Definition 3.5 (Refinement)**.** A partition $\tilde{P}$ is said to be a refinement of a partition P if $P \subset \tilde{P}$.

Definition 3.6 (Common refinement)**.** Given two partitions P_1 and P_2 of an interval $[a, b]$, we say that $\tilde{P}$ is their common refinement if $P_1 \cup P_2 \subset \tilde{P}$.

Definition 3.7 (Darboux sums)**.** Given a partition P of $[a, b]$ and a bounded function f, define

$$m_j = \inf_{t\in[x_j, x_{j+1}]} f(t), \qquad M_j = \sup_{t\in[x_j, x_{j+1}]} f(t).$$

The lower and upper Darboux sums are defined as

$$L(f, P) = \sum_{j=0}^{n-1} m_j(x_{j+1} - x_j), \qquad U(f, P) = \sum_{j=0}^{n-1} M_j(x_{j+1} - x_j).$$

Lemma 3.4. *If P is a partition of an interval $[a, b]$, then*

$$L(f, P) \leq U(f, P).$$

Proof. This follows directly from the fact that the infimum is less than or equal to the supremum, i.e. $m_j \leq M_j$. □

Theorem 3.5. *Let $\tilde{P}$ be a refinement of a partition P of an interval $[a, b]$. Then,*

$$L(f, P) \leq L(f, \tilde{P}) \leq U(f, \tilde{P}) \leq U(f, P)$$

Proof. Suppose that

$$P = \{x_0, \dots, x_k, x_{k+1}, \dots, x_n\},$$
$$\tilde{P} = \{x_0, \dots, x_k, y, x_{k+1}, \dots, x_n\}.$$

Set

$$m_1 = \inf_{t \in [x_k, y]} f(t), \qquad m_2 = \inf_{t \in [y, x_{k+1}]} f(t), \qquad m = \inf_{t \in [x_k, x_{k+1}]} f(t).$$

Then, observe that

$$L(f, \tilde{P}) - L(f, P) = m_1(y - x_k) + m_2(x_{k+1} - y) - m(x_{k+1} - x_k).$$

Now, from the properties of the infimum, $m_1 \geq m$ and $m_2 \geq m$, so

$$L(f, \tilde{P}) - L(f, P) \geq m(y - x_k + x_{k+1} - y - x_{k+1} + x_k) = 0.$$

This procedure of adding one point can be repeated finitely many times to obtain the same conclusion for any refinement of P. The case for the upper sum is analogous. □

Corollary 3.5.1. *For any two partitions P_1 and P_2 of $[a, b]$,*

$$L(f, P_1) \leq U(f, P_2).$$

Proof. Note that $P_1 \cup P_2$ is a common refinement of P_1 and P_2, hence

$$L(f, P_1) \leq L(f, P_1 \cup P_2) \leq U(f, P_1 \cup P_2) \leq U(f, P_2).$$ □

Corollary 3.5.2. *If $\{P_n\}$ is a sequence of refinements of a partition P_0 of $[a, b]$, then the following limits exist.*

$$L_{f,P_n} = \lim_{n \to \infty} L(f, P_n), \qquad U_{f,P_n} = \lim_{n \to \infty} U(f, P_n).$$

Proof. This follows from the monotone convergence theorem, together with the fact that $U(f, P_0)$ and $L(f, P_0)$ are upper and lower bounds of the two respective sequences. □

Corollary 3.5.3. *The following quantities exist, where the infimum and supremum is taken over all possible partitions P of $[a, b]$.*

$$L_f = \sup L(f, P), \qquad U_f = \inf U(f, P).$$

Furthermore, for any partition P,

$$L(f, P) \leq L_f \leq U_f \leq U(f, P).$$

Proof. First examine the set of all lower Darboux sums, $\{L(f, P)\}$. This set is non-empty, since any partition of $[a, b]$ gives a corresponding lower sum. Note that we have already demanded that f is bounded! This set is also bounded above, by any upper sum. Thus, the completeness of the reals guaranteed the existence of a supremum. The case for upper sums is analogous.

The outermost inequalities trivially follow from the definitions of the infimum and supremum. The middle inequality follows from the fact that if A and B are two subsets of $\mathbb{R}$ such that $\alpha \in A, \beta \in B$ implies $\alpha \leq \beta$, then $\sup A \leq \inf B$. □

Definition 3.8 (Darboux integrals)**.** The lower and upper Darboux integrals of a function f are defined as

$$L_f = \sup L(f, P), \qquad U_f = \inf U(f, P).$$

Here, the infimum and supremum is taken over all possible partitions P of $[a, b]$.

If $L_f = U_f$, then the common integral is simply called the Darboux integral,

$$\int_a^b f = L_f = U_f.$$

Such a function f is called Darboux integrable.

Theorem 3.6. *A function f is Darboux integrable on $[a, b]$ if and only if for every $\epsilon > 0$, there exists a partition P such that*

$$U(f, P) - L(f, P) < \epsilon.$$

Proof. First, assume that given $\epsilon > 0$, there exists a partition P such that

$$U(f, P) - L(f, P) < \epsilon.$$

By the previous corollary,

$$U_f - L_f \leq U(f, P) - L(f, P) < \epsilon$$

for all > 0, so $U_f = L_f$ giving Darboux integrability.

Now, suppose that f is Darboux integrable on $[a, b]$. This means that $U_f = L_f$. Using the definitions of supremum and infimum, for $\epsilon > 0$, there exists a partition P_1 such that $U(f, P_1) - U_f < \epsilon/2$ and a partition P_2 such that $L_f - L(f, P_2) < \epsilon/2$. Adding,

$$U(f, P_1) - L(f, P_2) < \epsilon.$$

Now, setting $P = P_1 \cup P_2$ as a common refinement of P_1 and P_2, we have

$$U(f, P) - L(f, P) < U(f, P_1) - L(f, P_2) < \epsilon.$$ □

Lemma 3.7. *Let f be bounded on $[a, b]$, and let P' be any partition of that interval. Then for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all partitions P with $\|P\| < \delta$,*

$$U(f, P) - U(f, P \cup P') < \epsilon.$$

Proof. Using the boundedness of f, choose $M \in \mathbb{R}$ such that $|f(x)| < M$ for all $x \in [a,b]$. Suppose that $P' = \{x_1, \dots, x_n\}$. Set $\delta = \epsilon/4nM$. Now, let P be any partition of $[a,b]$ with $\|P\| < \delta$. Write $P = \{y_i\}$, and $P \cup P' = \{z_i\}$ where all these sets are ordered. Now, note that if one of the subintervals $[z_j, z_{j+1}]$ does not contain a point x_i, then the term $M_j(y_{j+1} - y_j)$ cancels from $U(f,P) - U(f, P \cup P')$. Whenever there is an x_i in $[z_j, z_{j+1}]$, we have a term of the form

$$M_j(y_{j+1} - y_j) - M'(x_i - y_j) - M''(y_{j+1} - x_i),$$

where M', M'' are the supremums over the two pieces, each less than M. This means that this term is bounded by $4M\delta$. Since this can happen at most n times,

$$U(f,P) - U(f, P \cup P') < 4nM\delta = \epsilon. \qquad \square$$

Theorem 3.8. *Riemann and Darboux integrability are equivalent and assign the same value to the integrals.*

Proof. First assume that f is Riemann integrable on $[a,b]$. By Theorem 3.2, f is bounded so the Darboux upper and lower sums are well defined. Given $\epsilon > 0$, we seek a partition P such that

$$U(f,P) - L(f,P) < \epsilon.$$

Now, Riemann integrability guarantees the existence of a $\delta > 0$ such that for all tagged partitions $\dot{P}$ with $\|\dot{P}\| < \delta$,

$$|S(f,P) - \ell| < \frac{\epsilon}{3}$$

where ℓ is the value of the Riemann integral. Choose P with n subintervals. Now, let $\dot{P}_\xi$ be tagged with ξ_j and $\dot{P}_\zeta$ be tagged with ζ_j. From the definitions of the infimum and supremum, we choose our tags such that

$$f(\xi_j) - m_j < \frac{\epsilon}{6(b-a)}, \qquad M_j - f(\zeta_j) < \frac{\epsilon}{6(b-a)}.$$

This gives

$$M_j - m_j < f(\xi_j) - f(\zeta_j) + \frac{\epsilon}{3(b-a)}.$$

Thus,

$$\begin{aligned} U(f,P) - L(f,P) &< \sum_{j=0}^{n-1} \left(f(\xi_j) - f(\zeta_j) + \frac{\epsilon}{3(b-a)} \right) (x_{j+1} - x_j) \\ &< S(f, \dot{P}_\xi) - S(f, \dot{P}_\zeta) + \frac{\epsilon}{3(b-a)} \cdot (b-a) \\ &< |S(f, \dot{P}_\xi) - \ell| + |\ell - S(f, \dot{P}_\zeta)| + \frac{\epsilon}{3} \\ &< \epsilon. \end{aligned}$$

This proves that f is Darboux integrable on $[a,b]$, i.e. $U_f = L_f$. We now wish to show that $U_f = L_f = \ell$. Let $\epsilon > 0$. Using the properties of the infimum and supremum, we find partitions P_1, P_2 and $\dot{P}_3$ such that

$$L_f - L(f, P_1) < \frac{\epsilon}{6}, \qquad U(f, P_2) - U_f < \frac{\epsilon}{6}, \qquad |S(f, \dot{P}_3) - \ell| < \frac{\epsilon}{3}.$$

Setting $P = P_1 \cup P_2 \cup P_3$,

$$L_f - L(f, P) < \frac{\epsilon}{6}, \qquad U(f, P) - U_f < \frac{\epsilon}{6}, \qquad |S(f, \dot{P}) - \ell| < \frac{\epsilon}{3}.$$

Now,
$$L(f,P) \le S(f,\dot{P}) \le U(f,P) < L(f,P) + \frac{\epsilon}{3}.$$
This means that $S(f,\dot{P}) - L(f,P) < \epsilon/3$. Now,
$$|\ell - L_f| \le |\ell - S(f,\dot{P})| + |S(f,\dot{P}) - L(f,P)| + |L(f,P) - L_f| < \epsilon.$$
This forces $U_f = L_f = \ell$.

Now assume that f is Darboux integrable on $[a,b]$. This means that $U_f = L_f$. For $\epsilon > 0$, choose a partition P' such that
$$U(f,P') - U_f < \frac{\epsilon}{2}.$$
Set $\delta_1 = \epsilon/8nM$, and use our previous lemma to conclude that for any partition P of $[a,b]$ with $\|P\| < \delta_1$,
$$U(f,P) < U(f,P\cup P') + \frac{\epsilon}{2} \le U(f,P') + \frac{\epsilon}{2} < U_f + \epsilon.$$
Similarly, we can choose $\delta_2 > 0$ such that for all partitions P with $\|P\| < \delta_2$,
$$L(f,P) > L_f - \epsilon.$$
Setting $\delta = \min\{\delta_1, \delta_2\}$, we have
$$L_f - \epsilon < L(f,P) < S(f,\dot{P}) < U(f,P) < U_f + \epsilon.$$
Thus, for all tagged partitions $\dot{P}$ with $\|\dot{P}\| < \delta$, we have
$$|S(f,\dot{P}) - U_f| < \epsilon.$$
□

Theorem 3.9. *Any real continuous function on $[a,b]$ is Riemann integrable.*

Proof. Note that any continuous function on a compact interval is uniformly continuous. Thus, for $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in [a,b]$, we have
$$|f(x) - f(y)| < \frac{\epsilon}{b-a}.$$
Now, construct a partition of $[a,b]$ which divides the interval into equal subintervals of length $(b-a)/n$, where n is chosen such that $\|P\| < \delta$. This immediately gives
$$U(f,P) - L(f,P) = \sum_{j=0}^{n-1} (M_j - m_j) \cdot \frac{1}{n}(b-a) \le n \cdot \frac{\epsilon}{b-a} \cdot \frac{b-a}{n} = \epsilon.$$
□

Theorem 3.10. *Any bounded, monotone function on $[a,b]$ is Riemann integrable.*

Proof. Without loss of generality, suppose that f is monotonically increasing. Now, f on each interval attains its minimum and maximum at the endpoints, so
$$U(f,P) - L(f,P) = \sum_{j=0}^{n-1} (f(x_{j+1}) - f(x_j))(x_{j+1} - x_j).$$
Let $\epsilon > 0$. Choose P such that each subinterval has length $(b-a)/n$. Since f is bounded on $[a,b]$, we can choose n to be sufficiently large such that
$$U(f,P) - L(f,P) = (f(b) - f(a))\frac{b-a}{n} < \epsilon.$$
□

Theorem 3.11. *Let f be Riemann integrable on $[a, b]$, and let g be such that $g(x) = f(x)$ for all $x \in [a, b] \setminus \{c\}$, and $f(c) \neq g(c)$ for some $c \in [a, b]$. Then, g is also Riemann integrable on $[a, b]$.*

Proof. Let the integral of f on $[a, b]$ be ℓ. For $\epsilon > 0$, let $\delta > 0$ be such that for all tagged partitions $\dot{P}$ with $\|\dot{P}\| < \delta$, we have

$$|S(f, \dot{P}) - \ell| < \frac{\epsilon}{2}.$$

Now, note that

$$|S(f, \dot{P}) - S(g, \dot{P})| \leq 2\delta |f(c) - g(c)|.$$

This is because c can be a tag of at most 2 subintervals. Relabelling δ such that $2\delta|f(c) - g(c)| < \epsilon/2$, we see that for all tagged partitions $\dot{P}$ with $\|\dot{P}\| < \delta$,

$$|S(g, \dot{P}) - \ell| \leq |S(g, \dot{P}) - S(f, \dot{P})| + |S(f, \dot{P}) - \ell| < \epsilon.$$ □

Corollary 3.11.1. *Let f be Riemann integrable on $[a, b]$, and let g be such that $g(x) = f(x)$ on all but finitely many points in $[a, b]$. Then, g is also Riemann integrable on $[a, b]$.*

Theorem 3.12. *Let f be a bounded function with a single point of discontinuity in $[a, b]$. Then, f is Riemann integrable on $[a, b]$.*

Proof. Let $c \in [a, b]$ be the point of discontinuity. From the boundedness of f, choose M such that $|f(x)| < M$ for all $x \in [a, b]$. Now, for $\epsilon > 0$, choose a partition of $[a, b]$ such that the subinterval containing c has length at most $\epsilon/2M$. The continuity and boundedness of f on the remaining subintervals means that it is Riemann integrable on them, so we can repartition them with P' such that $U(f, P') - L(f, P') < \epsilon/2$. Thus, the difference in the upper and lower sums for the total partition P is bounded as

$$U(f, P) - L(f, P) < \frac{\epsilon}{2} + (M_i - m_i)\frac{\epsilon}{2M} \leq \epsilon.$$ □

Corollary 3.12.1. *Let f be a bounded function with finitely many points of discontinuity in $[a, b]$. Then, f is Riemann integrable on $[a, b]$.*

Corollary 3.12.2. *Let f be a bounded function whose points of discontinuities in $[a, b]$ have finitely many limit points. Then, f is Riemann integrable.*

Theorem 3.13. *The collection of all Riemann integrable functions on $[a, b]$ forms a vector space.*

Theorem 3.14. *Let f and g be Riemann integrable on $[a, b]$, such that*

$$f(x) \leq g(x)$$

for all $x \in [a, b]$. Then,

$$\int_a^b f \leq \int_a^b g.$$

Theorem 3.15. *Let f be Riemann integrable on $[a, b]$, and let $c \in [a, b]$. Then,*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Definition 3.9. Let f be Riemann integrable on $[a, b]$. We define

$$\int_a^b f = -\int_b^a f.$$

Theorem 3.16. *Let f be Riemann integrable on $[a, b]$, and let*

$$m \leq f(x) \leq M$$

for all $x \in [a, b]$. Also let $\varphi\colon [m, M] \to \mathbb{R}$ be continuous, and define $h = \varphi \circ f$ on $[a, b]$. Then, h is Riemann integrable on $[a, b]$.

Proof. Let $\epsilon > 0$. Note that φ must be uniformly continuous on $[a, b]$, which means that there exists $\delta > 0$ such that for all $x, y \in [m, M]$ with $|x - y| < \delta$, we have

$$|\varphi(x) - \varphi(y)| < \epsilon.$$

From the Riemann integrability of f on $[a, b]$, we find a partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ such that $\|P\| < \delta$ and

$$U(f, P) - L(f, P) < \delta^2.$$

We define M_j and m_j in the usual way, with respect to f and P. Also define M_j^* and m_j^* with respect to $h = \varphi \circ f$ and P. Consider the sets

$$A = \{j \in \{0, \dots, n-1\}\colon M_j - m_j < \delta\},$$

$$B = \{j \in \{0, \dots, n-1\}\colon M_j - m_j \geq \delta\}.$$

For each $j \in A$, we use the uniform continuity of φ to get

$$M_j^* - m_j^* < \epsilon.$$

When $k \in B$, let $M' = \sup_{[a,b]} h = \sup_{[m,M]} \varphi$, so

$$M_k^* - m_k^* < 2M'.$$

Now,

$$\delta \sum_{k\in B}(x_{k+1}-x_k) \leq \sum_{k\in B}(M_k - m_k)(x_{k+1}-x_k) \leq \delta^2,$$

where the first inequality is because $k \in B$, and the second is because of the Riemann integrability of f. This gives

$$\sum_{k\in B} x_{k+1} - x_k < \delta.$$

Thus,

$$U(h,P) - U(h,P) = \sum_{j\in A}(M_j^* - m_j^*)\Delta x + \sum_{k\in B}(M_k^* - m_k^*)\Delta x \leq \epsilon(b-a) + 2M'\delta.$$

Choose $\delta < \epsilon$, whence

$$U(h,P) - L(h,P) \leq \epsilon(a - b + 2M'). \qquad \square$$

Corollary 3.16.1. *Let f be Riemann integrable on $[a,b]$. Then, f^2 is also Riemann integrable on $[a,b]$.*

Corollary 3.16.2. *Let f and g be Riemann integrable on $[a,b]$. Then, fg is also Riemann integrable on $[a,b]$.*

Proof. Use

$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]. \qquad \square$$

Theorem 3.17 (Lebesgue-Vitali theorem)**.** *A bounded function on a compact interval $[a,b]$ is Riemann integrable if and only if it is continuous almost everywhere, i.e. the set of its points of discontinuity has measure zero.*

Remark. A set $\mathscr{D} \subset \mathbb{R}$ has Lebesgue measure zero if for every $\epsilon > 0$, there exists countable collection of open intervals $\{\mathscr{I}_n\}$ such that

$$\mathscr{D} \subseteq \bigcup_{n=1}^{\infty} \mathscr{I}_n, \qquad \sum_{n=1}^{\infty} \mu(\mathscr{I}_n) < \epsilon,$$

where $\mu(\mathscr{I}_n) = b_n - a_n$ is the length of each open interval $\mathscr{I}_n = (a_n, b_n)$.

Example. The Dirichlet function, defined as

$$f\colon [0,1] \to \mathbb{R}, \qquad f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}, \end{cases}$$

is not Riemann integrable since its set of discontinuities is the entire interval $[0,1]$.

Lemma 3.18. *Any countable set has Lebesgue measure zero.*

Proof. Enclose each element of the set with an open interval of length $\epsilon/2^n$, for $n = 1, 2, \dots$. □

Theorem 3.19. *Let f be Riemann integrable on $[a, b]$ and let $g\colon [a, b] \to \mathbb{R}$ be defined as*

$$g(x) = \int_a^x f(t)\, dt.$$

Then,

1. *g is continuous on $[a, b]$.*
2. *If f is continuous at some $x_0 \in (a, b)$, then g is differentiable at x_0 and $g'(x_0) = f(x_0)$.*

Proof. To show that g is continuous, first note that f is Riemann integrable so it must be bounded, i.e. we find $M > 0$ such that $|f(x)| < M$ on $[a, b]$. Now, for $x, y \in [a, b]$, $x > y$, note that

$$|g(x) - g(y)| = |\int_y^x f(t)\, dt| \leq M(x - y).$$

Therefore, for $\epsilon > 0$, set $\delta = \epsilon/M$ whereby for all $x, y \in [a, b]$ with $|x - y| < \delta$, we have

$$|g(x) - g(y)| \leq M|x - y| < \epsilon.$$

This gives the continuity of g.
Now, suppose that f is continuous at some $x_0 \in (a, b)$. Thus, given $\epsilon > 0$, we can choose $\delta > 0$ such that $|x - x_0| < \delta$ implies that $|f(x) - f(x_0)| < \epsilon$. Choose $0 < h < \delta$, whence

$$g(x_0 + h) - g(x_0) = \int_{x_0}^{x_0+h} f(t)\, dt.$$

Rearranging, we can write

$$\frac{g(x_0 + h) - g(x_0)}{h} - f(x_0) = \frac{1}{h} \int_{x_0}^{x_0+h} f(t) - f(x_0)\, dt.$$

Taking absolute values, we see that

$$\left| \frac{g(x_0 + h) - g(x_0)}{h} - f(x_0) \right| \leq \frac{1}{h} \int_{x_0}^{x_0+h} |f(t) - f(x_0)|\, dt \leq \frac{\epsilon h}{h} = \epsilon.$$

The case for $-\delta < h < 0$ is analogous. This gives

$$g'(x_0) = \lim_{h \to 0} \frac{g(x_0 + h) - g(x_0)}{h} = f(x_0).$$ □

Theorem 3.20 (Fundamental Theorem of Calculus)**.** *Let f be Riemann integrable on $[a, b]$. Suppose that g is continuous on $[a, b]$, differentiable on (a, b), and satisfies $g'(x) = f(x)$ for all $x \in (a, b)$. Then,*

$$g(b) - g(a) = \int_a^b f(x)\, dx.$$

Proof. Since f is Riemann integrable on $[a,b]$, given $\epsilon > 0$, there exists a $\delta > 0$ such that for all tagged partitions $\dot{P}$ with $\|\dot{P}\| < \delta$,

$$|S(f, \dot{P}) - \ell| < \epsilon,$$

where ℓ is the integral of f on $[a,b]$. Choose one such partition $\dot{P}$. Now, pick a subinterval $[x_j, x_{j+1}]$. Note that g is continuous on this closed interval, and differentiable on the open interval. Using the Mean Value Theorem, we choose $\xi_j \in (x_j, x_{j+1})$ such that

$$g(x_{j+1}) - g(x_j) = g'(\xi_j)(x_{j+1} - x_j) = f(\xi_j)(x_{j+1} - x_j).$$

Choosing such ξ_j for all $j = 0, \dots, n-1$, we have

$$\sum_{j=0}^{n-1} g(x_{j+1}) - g(x_j) = \sum_{j=0}^{n-1} f(\xi_j)(x_{j+1} - x_j) = S(f, \dot{P}_\xi),$$

where $\dot{P}_\xi$ denotes the use of the tags ξ_j. Note that $\|\dot{P}\| = \|\dot{P}_\xi\| < \delta$. Also, the first sum telescopes to $g(b) - g(a)$. Thus,

$$|g(b) - g(a) - \ell| < \epsilon$$

for all $\epsilon > 0$, which gives the desired equality. □

Theorem 3.21 (Integration by parts)**.** *Let f and g be continuous on $[a,b]$ and differentiable on (a,b). Also let f' and g' be Riemann integrable on $[a,b]$. Then,*

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx.$$

Proof. The proof involves defining $h = f \cdot g$, and using the Fundamental Theorem of Calculus on

$$h' = f'g + fg'.$$ □

Theorem 3.22 (Substitution of variables)**.** *Let f be Riemann integrable on $[a,b]$ and let $\varphi\colon [c,d] \to [a,b]$ be a surjective, strictly increasing map such that φ is differentiable on (c,d). Then,*

$$\int_a^b f(x)\,dx = \int_c^d (f \circ \varphi)(x)\,\varphi'(x)\,dx.$$

Theorem 3.23 (Uniform convergence theorem)**.** *Let $\{f_n\}$ be a sequence of Riemann integrable functions on $[a,b]$ such that $f_n \to f$ uniformly on $[a,b]$. Then, f is also Riemann integrable on $[a,b]$, and*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

Theorem 3.24 (Weierstrass approximation theorem)**.** *Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then, there exists a sequence of polynomials $\{p_n\}$ such that $p_n \to f$ uniformly on $[a,b]$.*

Proof. Without loss of generality, we may only consider continuous functions on $[0,1]$ such that $f(0) = f(1) = 0$. Also define $f(x) = 0$ outside $[0,1]$. Thus, f is uniformly continuous on $\mathbb{R}$. Define the polynomials

$$q_n(x) = c_n(1-x^2)^n,$$

where we choose c_n such that

$$\int_{-1}^{+1} q_n(x)\,dx = 1.$$

Since Q_n is even, compute

$$\int_{-1}^{+1} (1-x^2)^n\,dx = 2\int_0^1 (1-x^2)^n\,dx$$

This can be split into

$$2\left[\int_0^{1/\sqrt{n}} (1-x^2)^n\,dx + \int_{1/\sqrt{n}}^1 (1-x^2)^n\,dx\right] \geq 2\int_0^{1/\sqrt{n}} (1-x^2)^n\,dx.$$

Using Bernoulli's inequality,

$$\int_{-1}^{+1} (1-x^2)^n\,dx \geq 2\int_0^{1/\sqrt{n}} (1-nx^2)\,dx = \frac{4}{3\sqrt{n}} \geq \frac{1}{\sqrt{n}}.$$

Hence, $c_n < \sqrt{n}$. This means that $q_n \to 0$ uniformly on any $[\delta, 1] \subset [0,1]$.

$$q_n(x) < \sqrt{n}(1-\delta^2)^n \to 0.$$

Now define

$$p_n(x) = \int_{-1}^{+1} f(x+t)q_n(t)\,dt.$$

Again, split this integral apart on the intervals $[-1,-x]$, $[-x,1-x]$, and $[1-x,1]$. The integrals on the outermost intervals vanish simply because $f(x) = 0$ on them. Hence,

$$p_n(x) = \int_{-x}^{1-x} f(x+t)q_n(t)\,dx.$$

Changing variables,

$$p_n(x) = \int_0^1 f(t)q_n(t-x)\,dt.$$

Now, note that p_n is a polynomial in x. This is because $q_n(t-x)$ is a polynomial in x, and our integral is over t. We claim that $p_n \to f$ uniformly on $[0,1]$. To show this, let $\epsilon > 1$, and use the uniform continuity of f to find $\delta > 0$ such that for all $|x-y| < \delta$, we have

$$|f(x) - f(y)| < \frac{\epsilon}{2}.$$

Now, compute

$$|p_n(x) - f(x)| = |\int_{-1}^{+1} f(x+t)q_n(t)\,dt - \int_{-1}^{+1} f(x)q_n(t)\,dt| \leq \int_{-1}^{+1} |f(x+t) - f(x)|q_n(t)\,dt.$$

Again, split this integral on the intervals $[-1, -\delta]$, $[-\delta, +\delta]$, and $[+\delta, 1]$. For the central integral, apply the uniform continuity argument to get the bound

$$I_2 \leq \frac{\epsilon}{2} \int_{-\delta}^{+\delta} q_n(t)\, dt < \frac{\epsilon}{2}.$$

Set $M = \sup |f(x)|$. Then, the outermost integrals are bounded as

$$I_1 \leq 2M \int_{-1}^{-\delta} q_n(t)\, dt, \qquad I_3 \leq 2M \int_{+\delta}^{+1} q_n(t)\, dt.$$

Use the fact that $q_n(x) < \sqrt{n}(1 - \delta^2)^n$ and take sufficiently large n to conclude

$$|p_n(x) - f(x)| < 4M\sqrt{n}(1 - \delta^2)^n + \frac{\epsilon}{2} < \epsilon.$$

□

Definition 3.10 (Improper integrals)**.** Let f be Riemann integrable on all $[a, c]$ such that $c < b$, where we allow $b = \infty$. Define

$$\int_0^b f(x)\, dx = \lim_{c \to b} \int_a^c f(x)\, dx,$$

provided that the limit exists and is finite. We say that the integral is convergent. Similarly, if

$$\lim_{c \to b} \int_a^c |f(x)|\, dx$$

exists and is finite, we say that the integral is absolutely convergent.

Theorem 3.25. *Let f be Riemann integrable on $[a, t]$ for every $t \geq a$. Assume that there is a positive constant M such that*

$$\int_a^t |f(x)|\, dx < M$$

for every $t \geq a$. Then, both of the following improper integrals exist

$$\int_a^\infty f(x)\, dx, \qquad \int_a^\infty |f(x)|\, dx.$$

Proof. Define

$$F\colon [a, \infty) \to \mathbb{R}, \qquad F(t) = \int_a^t |f(x)|\, dx.$$

Note that F is an increasing function, and $|F(t)| < M$ for all $t \geq a$. Hence, $\lim_{t\to\infty} F(t)$ exists, which gives the absolute convergence of the improper integral. This in turn gives the convergence of the improper integral. □

Theorem 3.26. *Let f be continuous, monotonically decreasing on $[0, \infty)$, and non-negative. Then, the improper integral*

$$\int_0^\infty f(x)\,dx$$

converges if and only if the series

$$\sum_{n=0}^\infty f(n)$$

converges.

Theorem 3.27. *Let f and g be positive functions and Riemann integrable on $[a, n]$ for all $n \in \mathbb{N}$, and let*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = A > 0.$$

Then, the improper integral

$$\int_a^\infty f(x)\,dx$$

converges if the improper integral

$$\int_a^\infty g(x)\,dx$$

converges.

Theorem 3.28. *Let G be bounded, and f be such that $f(x) \to 0$ as $x \to \infty$, and*

$$\int_0^\infty |f'(x)|\,dx < \infty.$$

Then, the improper integral

$$\int_0^\infty f'(x)G(x)\,dx$$

converges.