MA 2201 : Analysis II

Satvik Saha, 19MS154 April 11, 2021

Solution 1. Let f be monotonic on $[a, b]$. First, note that f must be bounded on $[a, b]$. If f is monotonically increasing, then we must have

$$f(a) \leq f(x) \leq f(b)$$

for all $a \leq x \leq b$. Similarly, if f is monotonically decreasing, then

$$f(b) \leq f(x) \leq f(a)$$

for all $a \leq x \leq b$.

Without loss of generality, let f be monotonically increasing on $[a, b]$, and let $M > 0$ such that $|f(x)| < M$ for all $x \in [a, b]$. Given $\epsilon > 0$, we wish to construct a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon,$$

where $U(f, P)$ and $L(f, P)$ are the upper and lower Darboux sums over the partition P. Now, note that over any subinterval $[x_j, x_{j+1}]$ of $[a, b]$, we have

$$m_j = \inf_{x \in [x_j, x_{j+1}]} f(x) = f(x_j), \qquad M_j = \sup_{x \in [x_j, x_{j+1}]} f(x) = f(x_{j+1}).$$

This is simply a consequence of the monotonicity of f – the maximum and minimum (which are the same as the supremum and infimum doe to boundedness) are attained at the endpoints. With this, set $\delta = (b - a)/n$ and let n be sufficiently large so that $(f(b) - f(a))\delta < \epsilon$, i.e.

$$n\epsilon > (f(b) - f(a))(b - a).$$

This can be done using the Archimedean property of the reals. Now, let the partition P be such that $[a, b]$ is divided into n equal subintervals, i.e. $P = \{x_0, x_1, \dots, x_n\}$ where

$$x_j = a + \frac{j}{n}(b - a).$$

Now, $x_j - x_{j+1} = \delta$, so

$$U(f, P) - L(f, P) = \sum_{j=0}^{n-1} (M_j - m_j)\delta = \delta \sum_{j=0}^{n-1} f(x_{j+1}) - f(x_j) = \delta \cdot (f(b) - f(a)) < \epsilon.$$

We have used the fact that the sum telescopes. Hence, f is Riemann integrable on $[a, b]$. The proof for monotonically decreasing f is analogous, since $-f$ will be monotonically increasing and the negative of a Riemann integrable function is clearly Riemann integrable.

Lemma 1. *Any countable set (finite or countably infinite) has Lebesgue measure zero.*

Proof. Let S be countable, and let $\mathscr{J} \subseteq \mathbb{N}$ be the indices of S. Create the open intervals

$$\mathscr{O}_j = \left(x_j - \frac{\epsilon}{2^{j+1}}, x_j + \frac{\epsilon}{2^{j+1}}\right),$$

for all $x_j \in S$, $j \in \mathscr{J}$. Thus, $\mu(\mathscr{O}_j) = \epsilon/2^j$, where μ is the length of the interval. Also, the union of all such $\mathscr{O}_j$, say $\mathscr{O}$, forms a cover of S. The length of this cover is bounded as

$$\mu(\mathscr{O}) \leq \sum_{j \in \mathscr{J}} \mu(\mathscr{O}_j) \leq \sum_{j=1}^{\infty} \frac{\epsilon}{2^j} = \epsilon.$$

Since ϵ was arbitrary, we conclude that S has Lebesgue measure zero. □

Solution 2. Given two continuous functions f and g on $[a, b]$, a point $c \in [a, b]$, and the function

$$h\colon [a,b] \to \mathbb{R}, \qquad h(x) = \begin{cases} f(x), & \text{if } x \in [a,c], \\ g(x), & \text{if } x \in (c,b). \end{cases}$$

Without loss of generality, define $h(b) = 0$ to cover the endpoint. Note that f and g are continuous, so h is continuous on $[a, c)$ and (c, b). This leaves potential discontinuities at the points c and b. Now, the set of discontinuities of h, say $\mathcal{D} \subseteq \{c, b\}$, is a finite set and hence has Lebesgue measure zero. Hence, by the Lebesgue-Vitali Theorem, h is Riemann integrable.

Alternative Define the step function

$$u\colon [a,b] \to \mathbb{R}, \qquad u(x) = \begin{cases} 0, & \text{if } x \in [a,c], \\ f(c) - g(c), & \text{if } x \in (c,b], \end{cases}$$

and define $h(b) = g(b)$ to cover the endpoint. Now, consider the function $h + u$. On the interval $[a, c)$, $h + u = f$ so it is continuous. On the interval $(c, b]$, $h + u = g - g(c) + f(c)$, which is again continuous. Now,

$$\lim_{x \to c^-} (h+u)(x) = \lim_{x \to c^-} f(x) = f(c),$$

and

$$\lim_{x \to c^+} (h+u)(x) = \lim_{x \to c^+} g(x) - g(c) + f(c) = f(c),$$

and $(h + u)(c) = f(c)$, so $h + u$ is continuous on the entirety of $[a, b]$, hence Riemann integrable. Thus, it suffices to show that the step function u is Riemann integrable, which would imply that the difference $h = (h + u) - u$ is Riemann integrable.

Let $f(c) - g(c) = \alpha$. Given $\epsilon > 0$, construct a partition $P = \{a, c - \delta, c + \delta, b\}$ where $\delta > 0$ is small enough to maintain the ordering of the points (we discard the trivial case where c is one of the endpoints a or b, which would mean that h is one of f or g). Furthermore, let δ be such that $2|\alpha|\delta < \epsilon$. Now, $U(f, P) - L(f, P)$ has contributions only from the central subinterval $[c - \delta, c + \delta]$, so

$$U(f,P) - L(f,P) = 2|\alpha|\delta < \epsilon.$$

Note that if f and g are two Riemann integrable functions on $[a, b]$, then it is possible to find two partitions P_1 and P_2 such that

$$U(f,P_1) - L(f,P_1) < \frac{\epsilon}{2}, \qquad U(g,P_2) - L(g,P_2) < \frac{\epsilon}{2}.$$

For the common refinement $P = P_1 \cup P_2$, we must have

$$U(f,P) - L(f,P) < \frac{\epsilon}{2}, \qquad U(g,P) - L(g,P) < \frac{\epsilon}{2}.$$

Adding,

$$U(f+g,P) - L(f+g,P) < \frac{\epsilon}{2}.$$

Also, note that

$$U(-f,P) = -L(f,P), \qquad L(-f,P) = -U(f,P)$$

because on any interval $[s, t]$,

$$\sup -f(x) = -\inf f(x), \qquad \inf -f(x) = -\sup f(x).$$

Hence,

$$U(-f,P) - L(-f,P) = U(f,P) - L(f,P) < \frac{\epsilon}{2}.$$

This gives the Riemann integrability of $f + g$ as well as $-f$.