MA 2201 : Analysis II

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Solution 1. We find the intervals on which the given function $f\colon \mathbb{R} \to \mathbb{R}$ is strictly increasing or decreasing.

(i)
$$f(x) = (x+a)^2.$$

Note that $f(x)$ is a polynomial, hence infinitely differentiable everywhere. Calculate

$$f'(x) = 2(x+a), \qquad f''(x) = 2, \qquad f'''(x) = f^{(4)}(x) = \cdots = 0.$$

Now, f is strictly increasing precisely when $f'(x) > 0$, i.e. on the interval $(-a, \infty)$. Also, f is strictly decreasing when $f'(x) < 0$, i.e. on the interval $(-\infty, -a)$. Since $-a$ is a point of minima, $f'(-a) = 0$ and $f''(-a) > 0$, with no other x such that $f(x) = f(-a) = 0$, we may also include the endpoint.

The function f is strictly increasing on $[-a, \infty)$ and strictly decreasing on $(-\infty, -a]$.

(ii)
$$f(x) = ax^2 + bx + c, \qquad a, b, c \neq 0.$$

Again,
$$f'(x) = 2ax + b = 2a(x - \alpha), \qquad f''(x) = 2a,$$

where $\alpha = -b/2a$. If $a > 0$, then f is strictly increasing on $[\alpha, \infty)$ and strictly decreasing on $(-\infty, \alpha]$. If $a < 0$, then f is strictly decreasing on $[\alpha, \infty)$ and strictly increasing on $(-\infty, \alpha]$.

(iii)
$$f(x) = (ax+b)^3 = a^3(x-\alpha)^3, \qquad a, b \neq 0$$

where $\alpha = -b/a$. If $a > 0$, f is strictly increasing everywhere, and if $a < 0$, then f is strictly decreasing everywhere.

This is easily seen, because $s < t \implies s^3 < t^3$ for all $s, t \in \mathbb{R}$. Thus, $(s-\alpha)^3 < (t-\alpha)^3$ so when $a > 0$, $f(s) < f(t)$ and when $a < 0$, $f(s) > f(t)$.

Solution 2. We find the Taylor expansion of order k of the following functions.

(i)
$$f(x) = \frac{1}{1+x^2}.$$

To expand about $x = 0$, we compute

$$f(x) = \frac{1}{(1+ix)(1-ix)} = \frac{1}{2}\left[\frac{1}{1+ix} + \frac{1}{1-ix}\right].$$

Since
$$\frac{d^n}{dx^n}\frac{1}{1+ax} = \frac{(-1)^n a^n n!}{(1+ax)^{n+1}},$$

we write
$$f^{(n)}(x) = \frac{(-1)^n n!}{2}\left[\frac{i^n}{(1+ix)^{n+1}} + \frac{(-i)^n}{(1-ix)^{n+1}}\right].$$

At $x = 0$,
$$f^{(n)}(0) = \frac{1}{2} i^n (-1)^n n! \left[1 + (-1)^n\right].$$

When n is odd, $f^{(n)}(0) = 0$. When n is even, we have $f^{(n)}(0) = i^n n!$. Thus,

$$\begin{aligned} f(x) &= \sum_{n=0}^{k} \frac{f^{(n)}(0)x^n}{n!} + O(x^{k+1}) \\ &= 1 - x^2 + x^4 - \cdots + (-1)^{m/2} x^m + O(x^{k+1}), \end{aligned}$$

where m is the highest even integer less than or equal to k.

Alternatively: Write $1/(1+x^2)$ as a geometric power series,

$$f(x) = 1 - x^2 + x^4 - \cdots = \sum_{n=0}^{\infty} a_n x^n,$$

where $a_{4n} = 1$, $a_{4n+1} = 0$, $a_{4n+2} = -1$ and $a_{4n+3} = 0$. This converges absolutely and uniformly on any compact subinterval of $(-1, +1)$, simply because it is a geometric series with common ratio $-x^2$, whose absolute value $|x^2| < 1$ on the interval. As a corollary of Abel's Lemma [1], this power series is infinitely differentiable on its interval of convergence. Recall that

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Repeating this finitely many times,

$$f^{(m)}(x) = \sum_{n=m}^{\infty} n(n-1)\dots(n-m+1) a_n x^{n-m}.$$

Thus, we have $f^{(n)}(0) = n! a_n$, which vanishes for odd n and is alternately $\pm n!$ for even n. This gives us back our result

$$f(x) = 1 - x^2 + x^4 - \cdots + (-1)^{m/2} x^m + O(x^{k+1}),$$

where m is the highest even integer less than or equal to k.

(ii)

$$f(x) = \frac{1}{x}.$$

To expand about $x = 1$, we compute

$$f'(x) = -\frac{1}{x^2}, \qquad f''(x) = \frac{2}{x^3}, \qquad f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}.$$

Thus,

$$\begin{aligned} f(x) &= \sum_{n=0}^{k} \frac{f^{(n)}(1)(x-1)^n}{n!} + O(x^{k+1}) \\ &= 1 - (x-1) + (x-1)^n - \cdots + (-1)^k (x-1)^k + O(x^{k+1}). \end{aligned}$$

(iii)

$$f(x) = e^x.$$

To expand about $x = 0$, note that

$$f^{(n)}(x) = f(x) = e^x,$$

so

$$\begin{aligned} f(x) &= \sum_{n=0}^{k} \frac{f^{(n)}(0)x^n}{n!} + O(x^{k+1}) \\ &= 1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{k!}x^k + O(x^{k+1}). \end{aligned}$$

[1] In notes on differentiation, Corollary 2.15.1.

(iv)

$$f(x) = \sin x.$$

To expand about $x = 0$, note that

$$f'(x) = \cos x, \qquad f''(x) = -\sin x, \qquad f'''(x) = -\cos x, \qquad \ldots$$

In general,

$$f^{(4n\pm1)}(x) = \pm\cos x, \qquad f^{(4n+1\pm1)}(x) = \mp\sin x.$$

Thus, $f^{(2n)}(0) = 0$ and $f^{2n+1}(0) = (-1)^n$. Setting $m = 2\ell + 1$ equal to the greatest odd integer less than or equal to k, we have

$$\begin{aligned} f(x) &= \sum_{n=0}^{\ell} \frac{f^{(2n+1)}(0)x^{2n+1}}{(2n+1)!} + O(x^{k+1}) \\ &= \sum_{n=0}^{\ell} \frac{(-1)^n x^{2n+1}}{(2n+1)!} + O(x^{k+1}) \\ &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + \frac{(-1)^\ell}{m!}x^m + O(x^{k+1}). \end{aligned}$$

(v)

$$f(x) = x^k|x|.$$

First note that $x|x|$ is differentiable at the origin. The limit

$$\lim_{h\to0} \frac{h|h| - 0}{h} = \lim_{h\to0} |h| = 0$$

is well defined and exists, equal to zero. Hence, the product with x^{k-1} is also differentiable. Also note that for $x > 0$

$$\lim_{h\to0} \frac{(x+h)|x+h| - x|x|}{h} = \lim_{h\to0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to0} 2x + h = 2x,$$

and for $x < 0$,

$$\lim_{h\to0} \frac{(x+h)|x+h| - x|x|}{h} = \lim_{h\to0} \frac{-(x+h)^2 + x^2}{h} = \lim_{h\to0} -2x - h = -2x$$

hence the derivative of $x|x|$ is $2|x|$ everywhere. Using the product rule, compute

$$f'(x) = \frac{d}{dx}(x^{k-1} \cdot x|x|) = (k-1)x^{k-2} \cdot x|x| + 2x^{k-1}|x| = (k+1)x^{k-1}|x|.$$

This is of the same form as the original function, so we know that f' is further differentiable. Each time the differential operator acts on f, a factor is pulled in front and the power of x reduces by 1. After k such operations,

$$f^{(k)}(x) = (k+1) \cdot k \cdots 2 \cdot |x|.$$

In all cases, $f^{(n)}(0) = 0$. Thus, the k order Taylor polynomial about $x = 0$ is

$$\sum_{n=0}^{k} \frac{f^{(n)}(0)x^n}{n!} = 0.$$

However, note that Taylor's theorem does not apply since the $k+1$ order derivative does not exist at 0.

If we'd stopped at a $k-1$ order series, then we do have a remainder term of the form $f^{(k)}(c) = (k+1)!|c|$, for some c between 0 and x.

(vi)

$$f(x) = \begin{cases} x^{k+1}\sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

The k order Taylor expansion about 0 does not exist, since such functions are not generally differentiable k times at $x = 0$. As a counterexample, set $k = 2$. Now,

$$f'(0) = \lim_{x\to 0} \frac{x^3\sin(1/x) - 0}{x - 0} = \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

Now, for $x \neq 0$, use the product rule to compute

$$f'(x) = 3x^2\sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right).$$

Thus, the limit

$$f''(0) := \lim_{x\to 0}\frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0} 3x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

does not exist.

In general, define

$$F_n(x) = \begin{cases} x^n\sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \qquad G_n(x) = \begin{cases} x^n\cos\left(\frac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

For $x \neq 0$, see that

$$F_n'(x) = nF_{n-1}(x) - G_{n-2}(x), \qquad G_n'(x) = nG_{n-1}(x) + F_{n-2}(x).$$

The fact that the minimum leading power of x drops by 2 each time means that we get at most $\lceil k/2 \rceil$ derivatives.

We will instead show that the $m = \lceil k/2 \rceil$ Taylor polynomial about $x = 0$ is identically zero, i.e.

$$f^{(n)}(0) = 0$$

for $n = 0, 1, \ldots, m$. More generally,

$$F_{k+1}^{(n)}(0) = G_{k+1}^{(n)}(0) = 0$$

for $n = 0, 1, \ldots, m$.

We show this by induction on k. First, for $k = 1$, $m = 1$, $F_2(0) = 0$ and

$$F_2'(0) = \lim_{x\to 0}\frac{x^2\sin(1/x) - 0}{x - 0} = \lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\sin x}{x} = 0.$$

Similarly, $G_2(0) = 0$ and

$$G_2'(0) = \lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0.$$

Now, suppose that our statement is true for all $1 \leq n \leq k$, i.e. the 0 to $m = \lceil n/2 \rceil$ order derivatives of F_{n+1} and G_{n+1} all vanish at $x = 0$. Now, $F_{k+2}(0) = 0$ and

$$F_{k+2}'(0) = \lim_{x\to 0}\frac{x^{k+2}\sin(1/x) - 0}{x - 0} = \lim_{x\to 0} x^{k+1}\sin\left(\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\sin x}{x^{k+1}} = 0,$$

and similarly, $G_{k+2}(0) = 0$ and

$$G_{k+2}'(0) = \lim_{x\to 0} x^{k+1}\cos\left(\frac{1}{x}\right) = 0.$$

We use our recurrence relation (for $x \neq 0$) together with the above two facts (at $x = 0$) to conclude that

$$F'_{k+2}(x) = nF_{k+1}(x) - G_k(x), \qquad G'_{k+2}(x) = nG_{k+1}(x) + F_k(x)$$

everywhere. Now, F'_{k+2} and G'_{k+2} are linear combinations of F_{k+1}, F_k, G_{k+1} and G_k. Hence, we are guaranteed $\lceil (k-1)/2 \rceil$ further derivatives at 0 by our induction hypothesis. Since all derivatives (up to $\lceil (k-1)/2 \rceil$) of these four functions at $x = 0$ are zero, all these derivatives of the linear combinations F'_{k+2} and G'_{k+2} at $x = 0$ must also be zero. This gives us $\lceil (k+1)/2 \rceil$ derivatives of F_{k+2} and G_{k+2}, all equal to zero at $x = 0$, which proves our statement by induction.

Hence, the $\lceil k/2 \rceil$ order Taylor polynomial of f around $x = 0$ is

$$\sum_{n=0}^{\lceil k/2 \rceil} \frac{f^{(n)}(0)x^n}{n!} = 0.$$

We cannot write a remainder term in the form of Lagrange or Cauchy, since the subsequent derivative does not exist.

(vii)

$$f(x) = \begin{cases} e^{1/x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Again, the k order Taylor expansion about 0 does not exist, since f is not differentiable at $x = 0$. Note that f is unbounded on any δ neighbourhood of 0, since there will always be a point $1/N < \delta$, hence for all $n \leq N$, $1/n \in (-\delta, +\delta)$ and $f(1/n) = e^{n^2} \to \infty$. Thus, the limit

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{1}{x} e^{1/x^2}$$

does not exist.

Consider instead the function

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Now, we compute

$$e^{1/x^2} > 1 + \frac{1}{x^2}, \qquad 0 < e^{-1/x^2} < \frac{1}{1 + 1/x^2} = \frac{x^2}{1 + x^2}.$$

Dividing by x and taking limits as $x \to 0$, the squeeze theorem gives

$$f'(0) = \lim_{x \to 0} \frac{1}{x} \cdot e^{-1/x^2} = 0.$$

Elsewhere, the chain rule gives

$$f'(x) = e^{-1/x^2} \cdot \frac{2}{x^3}, \qquad x^3 f'(x) = 2f(x).$$

For the second derivative,

$$e^{1/x^2} > 1 + \frac{1}{x^2} + \frac{1}{2x^4} + \frac{1}{6x^6}, \qquad 0 < e^{-1/x^2} < \frac{1}{1 + 1/6x^6} = \frac{6x^6}{1 + 6x^6}.$$

Using the squeeze theorem again,

$$f''(0) = \lim_{x \to 0} \frac{1}{x} \cdot \frac{2}{x^3} e^{-1/x^2} = 0.$$

Now, we differentiate

$$\frac{d}{dx} x^3 f'(x) = 2f'(x), \qquad 3x^2 f'(x) + x^3 f''(x) = 2f'(x),$$

which when rearranged gives

$$x^3 f''(x) = (2 - 3x^2) f'(x), \qquad x^6 f''(x) = 2(2 - 3x^2) f(x).$$

Thus,

$$f'''(0) = \lim_{x \to 0} \frac{1}{x} f''(x) = \lim_{x \to 0} \frac{2 - 3x^2}{x^3} \cdot \frac{2}{x^3} \cdot f(x).$$

Using

$$e^{1/x^2} > 1 + \frac{1}{n! x^{2n}}, \qquad 0 < e^{-1/x^2} < \frac{n! x^{2n}}{1 + n! x^{2n}},$$

for $n = 4$,

$$f'''(0) = \lim_{x \to 0} \frac{2(2 - 3x^2)}{x^6} e^{-1/x^2} \leq \lim_{x \to 0} \frac{2(2 - x^2)}{x^6} \cdot \frac{4! x^8}{1 + 4! x^8} = 0,$$

hence the squeeze theorem again gives $f'''(0) = 0$.

In general, when we differentiate our functional equation, we get the form

$$x^{3n} f^{(n)}(x) = p_n(x) f(x),$$

where $p_n(x)$ is a polynomial of degree at most $2n$. To show this by induction, note that $p_0(x) = 1$, $p_1(x) = 2$, $p_2(x) = 2(2 - 3x^2)$. If this holds for all $n \leq m$, then

$$x^{3m} f^{(m)}(x) = p_m(x) f(x),$$

which when differentiated gives

$$3m x^{3m-1} f^{(m)}(x) + x^{3m} f^{(m+1)}(x) = p_m'(x) f(x) + p_m(x) f'(x).$$

Substituting the formulae for $f^{(m)}(x)$ and $f'(x)$,

$$\frac{3m}{x} p_m(x) f(x) + x^{3m} f^{(m+1)}(x) = p_m'(x) f(x) + \frac{2}{x^3} p_m(x) f(x),$$

which when multiplied by x^3 and rearranged gives

$$x^{3(m+1)} f^{(m+1)}(x) = \left[(2 - 3m x^2) p_m(x) + x^3 p_m'(x)\right] f(x).$$

The polynomial in brackets has degree at most $2m + 2$. This proves the desired statement by induction, and also gives us a recurrence relation for $p_m(x)$.

Now we show that $f^{(n)}(0) = 0$. Suppose that this is true for all $n \leq m$, hence

$$f^{(m+1)}(0) = \lim_{x \to 0} \frac{1}{x} f^{(m)}(x) = \lim_{x \to 0} \frac{p_m(x)}{x^{3m+1}} e^{-1/x^2}.$$

Using our inequality,

$$f^{(m+1)}(0) \leq \lim_{x \to 0} \frac{p_m(x)}{x^{3m+1}} \frac{(3m)! x^{6m}}{1 + (3m)! x^{6m}} = \lim_{x \to 0} \frac{(3m)! x^{3m-1} p_m(x)}{1 + (3m)! x^{6m}} = 0.$$

Hence, $f^{(m+1)}(0) = 0$, which proves that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

This means that the Taylor polynomial of any degree about $x = 0$ is just the zero polynomial. The remainder term is of course the function itself, $f(x)$.