# MA 2201 : Analysis II

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**Lemma 1.** *If $f\colon (a,b) \to \mathbb{R}$ is differentiable, then it is also continuous.*

*Proof.* Let $c \in (a,b)$. On the domain, the limit $\lim_{x\to c}(x-c)$ exists (and is equal to zero). The differentiability of $f$ guarantees that the limit

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

also exists. This means that the limit of the product also exists, and this gives

$$\lim_{x\to c}(x-a)\cdot \lim_{x\to c}\frac{f(x)-f(c)}{x-c} = \lim_{x\to c}(x-c)\cdot\frac{f(x)-f(c)}{x-c} = \lim_{x\to c} f(x)-f(c).$$

Thus, for all $c \in (a,b)$,

$$0 = \lim_{x\to c} f(x) - f(c), \qquad \lim_{x\to c} f(x) = f(c)$$

which gives the continuity of $f$. □

**Lemma 2.** *If $f$ and $g$ are continuous, then so are $f+g$ and $fg$. The quotient $f/g$ is continuous wherever $g(x) \neq 0$.* [1]

**Solution 1.** Given $f, g$ differentiable on $\mathbb{R}$. Note that they must also be continuous, which means that the limits

$$\lim_{x\to c} f(x) = f(c), \qquad \lim_{x\to c} g(x) = g(c)$$

are well defined and exist. Differentiability guarantees that the limits

$$f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}, \qquad g'(c) = \lim_{x\to c}\frac{g(x)-g(c)}{x-c}$$

also exist.

(i) Let $h = f+g$. Note that $h$ is continuous. To show that $h$ is differentiable on $\mathbb{R}$ with $h' = f'+g'$, consider

$$\begin{aligned}\lim_{x\to c}\frac{h(x)-h(c)}{x-c} &= \lim_{x\to c}\frac{(f(x)+g(x))-(f(c)+g(c))}{x-c} \\ &= \lim_{x\to c}\frac{(f(x)-f(c))+(g(x)-g(c))}{x-c} \\ &= \lim_{x\to c}\frac{f(x)-f(c)}{x-c} + \lim_{x\to c}\frac{g(x)-g(c)}{x-c} \\ &= f'(c)+g'(c).\end{aligned}$$

The limit can be separated because the individual limits exist. This shows that $h$ is differentiable on $\mathbb{R}$.

(ii) Let $h = fg$. Again, $h$ is continuous. Consider the limit

$$\begin{aligned}\lim_{x\to c}\frac{h(x)-h(c)}{x-c} &= \lim_{x\to c}\frac{f(x)g(x)-f(c)g(c)}{x-c} \\ &= \lim_{x\to c}\frac{f(x)g(x)-f(c)g(x)+f(c)g(x)-f(c)g(c)}{x-c} \\ &= \lim_{x\to c} g(x)\cdot\lim_{x\to c}\frac{f(x)-f(c)}{x-c} + f(c)\lim_{x\to c}\frac{g(x)-g(c)}{x-c} \\ &= g(c)f'(c)+f(c)g'(c).\end{aligned}$$

This shows that $fg$ is differentiable on $\mathbb{R}$, with $(fg)' = f'g+fg'$.

[1]See results from the first semester.

(iii) We show that $1/g$ is differentiable on the domain where $g(x) \neq 0$, which in turn would imply the differentiability of $f/g$ by the product rule above. Let $h = 1/g$ on the domain where $g(x) \neq 0$. Note that $g$ is continuous, which means that for any $c$ in its domain such that $g(c) \neq 0$, there exists a non-empty neighbourhood $(c-\delta, c+\delta)$ of $c$ where $g(x) \neq 0$. This means that the limit

$$\lim_{x\to c} h(x) = \lim_{x\to c} \frac{1}{g(x)} = \frac{1}{g(c)}$$

is well defined and exists. Now,

$$\begin{aligned}
\lim_{x\to c} \frac{h(x)-h(c)}{x-c} &= \lim_{x\to c} \frac{1/g(x) - 1/g(c)}{x-c} \\
&= -\lim_{x\to c} \frac{g(x)-g(c)}{g(x)g(c)(x-c)} \\
&= -\lim_{x\to c} \frac{1}{g(x)g(c)} \lim_{x\to c} \frac{g(x)-g(c)}{x-c} \\
&= -\frac{1}{g(c)^2} \cdot g'(c).
\end{aligned}$$

Thus shows that $1/g$ is differentiable, with $(1/g)' = -g'/g^2$. Hence, $f/g$ is differentiable, with $(f/g)' = (f'g - fg')/g^2$.

**Solution 2.** Given $f$ and $g$ are continuous on $[a, b]$ and differentiable on $(a, b)$. Define the function

$$h = (g(b) - g(a)) \cdot f - (f(b) - f(a)) \cdot g$$

on the same domain, and note that $h$ must be continuous on $[a, b]$ and differentiable on $(a, b)$ since it is a linear combination of $f$ and $g$. Also,

$$h(a) = f(a)g(b) - g(a)f(b) = h(b).$$

Thus, Rolle's Theorem guarantees the existence of $c \in (a, b)$ such that $h'(c) = 0$, whence

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)).$$

**Solution 3.** Consider the function

$$f\colon [-1, +1] \to \mathbb{R}, \qquad f(x) = x^3.$$

This function is continuous, and differentiable on $(-1, +1)$ as it is the product of the differentiable identity functions. It is also strictly increasing; given $x > y$, note that $x - y > 0$ and they both cannot be zero, so

$$f(x) - f(y) = x^3 - y^3 = (x-y)(x^2 + xy + y^2) = (x-y)\left[\left(x + \frac{1}{2}y\right)^2 + \frac{3}{4}y^2\right] > 0.$$

On the other hand[2],

$$f'(x) = 3x^2, \qquad f'(0) = 0.$$

**Solution 4.** Consider the function

$$f\colon (0, 2) \to \mathbb{R}, \qquad f(x) = \begin{cases} x, & \text{if } x < 1, \\ x + 1, & \text{if } x \geq 1. \end{cases}$$

Note that this is strictly increasing on $(0, 2)$, yet it is not continuous at $x = 1$. Thus, it cannot be differentiable on $(0, 2)$.

[2] Apply the product rule twice on the identity functions: $Dx = 1$ and $Dx^2 = xDx + xDx = 2x$, so $Dx^3 = x^2Dx + xDx^2 = 3x^2$, where $Df(x) \equiv f'(x)$.