MA 2201 : Analysis II

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Lemma 1. *The series* $\sum_{n=1}^{\infty} 1/n^2$ *converges to a real number between* 1 *and* 2.

Proof. Note that the partial sums of the given series are monotonically increasing, since all terms are positive. Also,

$$\begin{aligned} S_n = \sum_{k=1}^{n} \frac{1}{k^2} &= 1 + \sum_{k=2}^{n} \frac{1}{k^2} \\ &< 1 + \sum_{k=2}^{n} \frac{1}{k(k-1)} \\ &= 1 + \sum_{k=2}^{n} \left[\frac{1}{k-1} - \frac{1}{k}\right] \\ &= 1 + 1 - \frac{1}{n} \\ &< 2. \end{aligned}$$

Thus, the sequence of partial sums of the given series is bounded above, which means that it must converge by the monotone convergence theorem. □

Lemma 2. *If a series* $\sum_{n=1}^{\infty} \alpha_n$ *converges absolutely, then it also converges ordinarily.*

Proof. Let $\epsilon > 0$. From the Cauchy criterion and the absolute convergence of $\sum_{n=1}^{\infty} \alpha_n$, we can choose $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$\left| \sum_{k=1}^{m} |\alpha_k| - \sum_{k=1}^{n} |\alpha_k| \right| = \sum_{k=n+1}^{m} |\alpha_k| < \epsilon.$$

Using the triangle inequality, we have

$$\left| \sum_{k=n+1}^{m} \alpha_k \right| \leq \sum_{k=n+1}^{m} |\alpha_k| < \epsilon,$$

which proves that $\sum_{n=1}^{\infty} \alpha_n$ converges by the Cauchy criterion. □

Solution 1. We have been given the functions

$$f_n \colon \mathbb{R} \setminus \{-1/n^2\} \to \mathbb{R}, f_n(x) = \frac{1}{1 + n^2 x},$$

and we wish to examine the convergence of the series of functions $\sum_{n=1}^{\infty} f_n$.

Pointwise and absolute convergence : Note that when $x = 0$, the pointwise sum is $\sum_{n=1}^{\infty} 1$, which clearly diverges. When $x > 0$, note that $f_n(x) > 0$ for all $n \in \mathbb{N}$ so the sequence of partial sums is monotonically increasing. Also, the series is bounded above since

$$\sum_{k=1}^{n} f_k(x) = \sum_{k=1}^{n} \frac{1}{1 + k^2 x} < \sum_{k=1}^{n} \frac{1}{k^2 x} \leq \frac{2}{x},$$

so $\sum_{n=1}^{\infty} f_n(x)$ converges by the monotone convergence theorem. Note that this convergence is absolute, since all terms are positive in any case.

Similarly, when $x < -1$, $1 + n^2x < 0$ which means that all $f_n(x)$ are negative, so the sequence of partial sums is monotonically decreasing. Now, the series is bounded below since

$$\sum_{k=1}^{n} f_k(x) = \sum_{k=1}^{n} \frac{1}{1+k^2x} \geq \sum_{k=1}^{n} \frac{1}{k^2+k^2x} \geq \frac{2}{1+x},$$

so $\sum_{n=1}^{\infty} f_n(x)$ converges by the monotone convergence theorem. Note that this convergence is absolute, since all terms are negative in any case, hence the absolute sum is simply the negative of the ordinary sum.

Note that the sum $\sum_{n=1}^{\infty} f_n(x)$ is not defined when $x \in S = \{-1, -1/2^2, -1/3^2, \dots\}$. Suppose that $-1 < x < 0$ and $x \notin S$. Using the Archimedean property, choose $N \in \mathbb{N}$ such that $N^2|x| > 2$, which means that $1 + n^2x < n^2x/2 < 0$ for all $n \geq N$. Now for all $n \geq N$, we have the tail of the series

$$\sum_{k=N}^{n} f_k(x) = \sum_{k=N}^{n} \frac{1}{1+k^2x} \geq \sum_{k=N}^{n} \frac{1}{k^2x/2} \geq \frac{4}{x},$$

so the tail $\sum_{n=N}^{\infty} f_n(x)$ converges by the monotone convergence theorem. Thus, the complete series $\sum_{n=1}^{\infty} f_n(x)$ must also converge, since the sum of the first $N-1$ terms is just a finite number. Note that we have shown that the tail of the series from the N^{th} term onwards converges absolutely, since all those terms are negative, hence the absolute sum is the negative of the ordinary sum. The first $N-1$ terms are positive, so the complete absolute series including them is also convergent.

Thus, setting $S_0 = S \cup \{0\} = \{0, -1, -1/2^2, -1/3^2, \dots\}$, we have shown that $\sum_{n=1}^{\infty} f_n$ is convergent pointwise and absolutely on $\mathbb{R} \setminus S_0$.

Uniform convergence : We show that $\sum_{n=1}^{\infty}$ converges uniformly on any subset of $\mathbb{R} \setminus S_0$ which does not have 0 as a limit point, i.e. the series converges uniformly on $(-\infty, a] \cup [b, \infty) \setminus S_0$, where $a < -1$, $b > 0$.

First, let $x \in [b, \infty)$ with $b > 0$. Then,

$$|f_n(x)| = \frac{1}{1+n^2x} < \frac{1}{n^2x} \leq \frac{1}{n^2b},$$

hence $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[b, \infty)$ by the Weierstrass M-test.

Again, if $x \in (\infty, a]$ with $a < 0$, for all $n \geq N$ such that $N^2|a| > 2$, we have $1 + n^2x < n^2x/2 \leq n^2a/2$, so

$$|f_n(x)| = \frac{1}{|1+n^2x|} \leq \frac{2}{n^2|a|}$$

hence the series converges uniformly by the Weierstrass M-test again.

Disproof of uniform convergence : Suppose that the series converges uniformly on $x \in (0, \infty)$. This means that for $\epsilon = 1/3$, we can choose $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\left| \sum_{n=1}^{N+1} f_n(x) - \sum_{n=1}^{N} f_n(x) \right| = |f_{N+1}(x)| < \epsilon = \frac{1}{3}$$

for all $x \in (0, \infty)$ by the Cauchy criterion. On the other hand,

$$|f_{N+1}(1/(N+1)^2)| = \frac{1}{2} > \frac{1}{3},$$

which is a contradiction.

More broadly, suppose that $\sum f_n$ converges uniformly on any set J_0 which has 0 as a limit point. Note that J_0 cannot contain 0. This means that we can choose $x_0 \in J_0$ such that $|x_0| < 1/(N+1)^2$. Now,

$$|f_{N+1}(x_0)| = \frac{1}{|1+(N+1)^2x_0|} \geq \frac{1}{1+(N+1)^2|x_0|} > \frac{1}{2} > \frac{1}{3},$$

which is again a contradiction.

Solution 2. We have been given functions f_n such that for some $M > 0$,

$$|f_n(x)| \leq M|x|^n$$

for all $n \in \mathbb{N}$. We wish to examine the convergence of the series $\sum_{n=1}^{\infty} f_n$.

The series need not converge at all when $|x| \geq 1$. We supply the example $f_n(x) = x^n$, $M = 1$. Note that when $|x| \geq 1$, the series $\sum_{n=1}^{\infty} x^n$ does not converge because it fails the Cauchy criterion (n^{th} term test). We have $|x|^n \to \infty$ when $|x| > 1$ and $|x|^n \to 1$ when $|x| = 1$, whereas we demand $f_n(x) \to 0$ for the series to converge. Thus, we need only look at the interval $(-1, +1)$.

Note that this same counterexample shows that $\sum_{n=1}^{\infty} f_n$ need not converge uniformly on $(-1, +1)$. If $\sum_{n=1}^{\infty} x^n$ converged uniformly on $(-1, +1)$, then for $\epsilon = 1/2$, we could choose $N \in \mathbb{N}$ such that for $n = N$, $m = N + 1$,

$$\left| \sum_{n=1}^{N+1} f_n(x) - \sum_{n=1}^{N} f_n(x) \right| = |f_{N+1}(x)| = |x|^{N+1} < \epsilon = \frac{1}{2}$$

by the Cauchy criterion. On the other hand, note that $x_0 = (1/2)^{1/(N+1)} \in (-1, +1)$, so we demand

$$|f_{N+1}(x_0)| = \frac{1}{2} < \frac{1}{2},$$

which is absurd.

The series does converge absolutely, hence pointwise on $(-1, +1)$. For fixed $x \in (-1, +1)$, the geometric series

$$\sum_{n=1}^{\infty} M|x|^n = \frac{M|x|}{1 - |x|}$$

converges. Since $0 \leq |f_n(x)| \leq M|x|^n$, the series $\sum_{n=1}^{\infty} |f_n(x)|$ must converge by the comparison test. This means that $\sum_{n=1}^{\infty} f_n$ converges absolutely on $(-1, +1)$, which in turn means that it converges ordinarily (pointwise) on the same.

The series converges pointwise, absolutely, and uniformly on the compact interval $[-a, +a]$ where $0 < a < 1$. This is because for $|x| < a < 1$,

$$|f_n(x)| \leq M|x|^n \leq Ma^n,$$

so setting $M_n = Ma^n$, the series

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} Ma^n = \frac{Ma}{1 - a}$$

is convergent, which gives the uniform convergence of both $\sum_{n=1}^{\infty} f_n$ and $\sum_{n=1}^{\infty} |f_n|$ by the Weierstrass M-test. This of course implies the pointwise convergence of the same.