# MA 2201 : Analysis II

Satvik Saha, 19MS154 January 22, 2021

We state and prove the following theorems in order to simplify the problems ahead.

**Lemma 1.** *The sequence $\{1/n\}$ converges on $[0, 1]$, and $1/n \to 0$.*

*Proof.* Let $\epsilon > 0$ be arbitrary. Using the Archimedean property of the reals, choose $N \in \mathbb{N}$ such that $N\epsilon > 1$. One such choice is given by $N = \lfloor 1/\epsilon \rfloor + 1$. Thus, for all $n \geq N$, we have

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

This shows that $1/n \to 0$. □

**Theorem 2.** *Let $f_n : X \to Y$ and let $f_n \to f$ pointwise on $X$. For all $n \in \mathbb{N}$, set*

$$M_n = \sup_{x \in X} |f_n(x) - f(x)|.$$

*Then, $\{f_n\}$ converges uniformly on $X$ to $f$ if and only if $M_n \to 0$.*

*Proof.* Suppose that $f_n \to f$ uniformly on $X$. Let $\epsilon > 0$ be arbitrary, and let $N \in \mathbb{N}$ be such that for all $n \geq N$ and $x \in X$,

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}.$$

This means that for all $n \geq N$,

$$M_n = \sup |f_n(x) - f(x)| \leq \frac{\epsilon}{2} < \epsilon.$$

Also note that all $M_n \geq 0$, since they are the supremums of non-negative quantities. This means that $M_n \to 0$, as desired.

Now suppose that $M_n \to 0$. This means that for arbitrary $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|M_n| = \sup |f_n(x) - f(x)| < \epsilon.$$

Now, from the properties of the supremum, we see that for all $n \geq N$ and $x \in X$,

$$|f_n(x) - f(x)| \leq \sup |f_n(x) - f(x)| < \epsilon.$$

This proves that $f_n \to f$ uniformly. □

**Theorem 3.** *Let the functions $f_n : X \to Y$ be continuous, and suppose that $f_n \to f$ uniformly on $X$. Then, $f$ is continuous on $X$.*

*Proof.* Let $\epsilon > 0$. We wish to show that $f$ is continuous at arbitrary $x_0 \in X$.
Since $f_n \to f$ uniformly on $X$, we find $N \in \mathbb{N}$ such that for all $x \in X$ and $n \geq N$, we have

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}.$$

In particular, this holds for $n = N$, and $x = x_0$.
The continuity of each $f_n$ on $X$ means $f_N$ is continuous on $X$ in particular, so we can find $\delta > 0$ such that whenever $|x - x_0| < \delta$, we have

$$|f_N(x) - f_N(x_0)| < \frac{\epsilon}{3}.$$

Putting these together, for every $x \in X$ such that $|x - x_0| < \delta$, we have

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

This means that $f$ is continuous at $x_0$ for arbitrary $x_0 \in X$, i.e. $f$ is continuous on $X$. □

**Corollary 3.1.** *Let the functions $f_n \colon X \to Y$ be continuous, and let $f_n \to f$ pointwise on $X \subseteq \mathbb{R}$. If $f$ is not continuous on $X$, then the sequence of functions $\{f_n\}$ does not converge uniformly on $X$.*

*Proof.* This is simply the contrapositive of Theorem 2. $\square$

**Solution 1.**

(i) We have been given the functions

$$f_n \colon [0, \infty) \to \mathbb{R}, \qquad f_n(x) = \frac{x}{x+n}.$$

We show that $f_n \to 0$ pointwise on $[0, \infty)$, where by 0 we mean the zero function. Let $x \in [0, \infty)$ be arbitrary, and let $\epsilon > 0$. We choose $N \in \mathbb{N}$ such that $N\epsilon > x$ using the Archimedean property of the reals. Since $x \geq 0$, whenever $n \geq N$, we have

$$\frac{x}{x+n} \leq \frac{x}{n} \leq \frac{x}{N} < \epsilon,$$

so $|f_n(x)| < \epsilon$. This means that $f_n(x) \to 0$ for all $x \in [0, \infty)$, hence $f_n \to 0$ pointwise on $[0, \infty)$. This also means that by restricting the domain, $f_n \to 0$ on $[0, a]$ for $a > 0$.

We now show that $\{f_n\}$ does not converge uniformly on $[0, \infty)$. Suppose to the contrary that it did. Let $N \in \mathbb{N}$ be such that for all $n \geq N$ and for all $x \in [0, \infty)$, we have

$$|f_n(x) - 0| = \frac{x}{x+n} < \epsilon = \frac{1}{3}.$$

This must hold in particular for $x = N \in [0, \infty)$ and $n = N$. Plugging this in, we demand

$$\frac{N}{N+N} = \frac{1}{2} < \frac{1}{3},$$

which is absurd.

On the other hand, $\{f_n\}$ does converge to 0 uniformly on $[0, a]$, where $a > 0$. Let $\epsilon > 0$ be arbitrary. Note that for $x \in [0, a]$, we have $x \leq a$. Now, choose $N \in \mathbb{N}$ such that $N\epsilon > a$. Thus, whenever $n \geq N$ and $x \in [0, a]$, we have

$$|f_n(x) - 0| = \frac{x}{x+n} \leq \frac{x}{n} \leq \frac{a}{n} \leq \frac{a}{N} < \epsilon.$$

This means that $f_n \to 0$ uniformly on $[0, a]$.

Note that what we have shown is that on $[0, a]$,

$$M_n = \sup |f_n(x) - 0| < \frac{a}{n} \to 0.$$

(ii) We have been given the functions

$$f_n \colon \mathbb{R} \to \mathbb{R}, \qquad f_n(x) = \frac{nx}{x + n^2x^2}.$$

Note that we assume $f_n(0) = n/(1 + n^2 0) = n$. We first show that the sequence of functions $\{f_n\}$ does not converge on $\mathbb{R}$. This is because the sequence $f_n(0) = n \to \infty$ diverges. The same argument shows why $\{f_n\}$ does not converge on $[0, \infty)$. Thus, the question of uniform convergence does not arise on these domains either.

We show that $f_n \to 0$ on $[a, \infty)$, where $a > 0$ by proving the even stronger statement of uniform convergence. Since $x \geq a$, we have

$$M_n = \sup |f_n(x) - 0| = \sup \frac{n}{1 + n^2x} \leq \frac{n}{1 + n^2a} \leq \frac{n}{n^2a} \to 0.$$

Thus, $M_n \to 0$, which means that $f_n \to 0$ uniformly on $[a, \infty)$.

Formally, given $\epsilon > 0$, we choose $N \in \mathbb{N}$ such that $Na\epsilon \geq 1$, whence for all $n \geq N$ and $x \in [a, \infty)$, we have

$$|f_n(x) - 0| = \frac{nx}{x + n^2x^2} \leq \frac{nx}{n^2x^2} \leq \frac{1}{na} \leq \frac{1}{Na} < \epsilon.$$

This gives the uniform convergence of $f_n \to 0$ on $[a, \infty)$.

**Alternate definition at the missing point** We instead define $f_n(0) = 0$, in which case we observe that $f_n(0) \to 0$. When $x > 0$, $f_n(x) = n/(1 + n^2 x) < 1/nx \to 0$. When $x \neq 0$, note that $f_n(-1/n^2)$ is undefined. Nevertheless, note that when $x \neq -1/n^2$,

$$f_n(x) = \frac{nx}{x + n^2 x^2} = \frac{1/n}{1/n^2 + x}.$$

Since $1/n^2 \to 0$, we have $1/n^2 + x \to x \neq 0$ and $1/n \to 0$, hence $f_n(x) \to 0$. This gives $f_n \to 0$ pointwise on $\mathbb{R}$ , which in turn means that $f_n \to 0$ on $[0, \infty)$.

We show that $\{f_n\}$ does not converge uniformly on $[0, \infty)$. Suppose to the contrary that $f_n \to 0$ uniformly. Pick $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in [0, \infty)$, we have

$$|f_n(x) - 0| = \frac{nx}{x + n^2 x^2} < \frac{1}{3}.$$

This ought to hold for $x = 1/N \in [0, \infty)$ and $n = N$ in particular. Thus,

$$f_N(1/N) = \frac{1}{1/N + 1} < \frac{1}{3},$$

which means $1/N + 1 > 3$, or $1/N > 2$, or $N < 1/2$ which is absurd. Thus, $\{f_n\}$ cannot converge uniformly on $[0, \infty)$, nor on $\mathbb{R}$.

The arguments for $f_n \to 0$ uniformly on $[a, \infty)$ where $a > 0$ remain identical to what we've shown in the previous definition of $f_n$.

(iii) We have been given the functions

$$f_n \colon [0, \infty) \to \mathbb{R}, \qquad f_n(x) = \frac{nx}{1 + nx}.$$

Note that for all $x \in [0, \infty)$, we have $1/n \to 0$ so $1/n + x \to x$, thus when $x > 0$ we can write

$$f_n(x) = \frac{nx}{1 + nx} = \frac{x}{1/n + x} \to 1.$$

When $x = 0$, $f_n(x) = 0$ so $f_n(0) \to 0$. Thus, setting

$$f \colon [0, \infty) \to \mathbb{R}, \qquad f(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases},$$

we have $f_n \to f$ pointwise on $[0, \infty)$. By restricting the domain of $f_n$, we have $f_n \to 1$ pointwise on $[a, \infty)$ where $a > 0$.

We see that $\{f_n\}$ does not converge uniformly on $[0, \infty)$, because each $f_n$ is continuous, being the ratio of continuous functions with a non-zero denominator. However, the limit function $f$ is discontinuous. To see this, note that $f(1/n) = 1$ for all $n \in \mathbb{N}$, yet $f(0) = 0 \neq 1 = \lim_{n\to\infty} f(1/n)$.

Another way to see this is to note that $f_n(1/n) = 1/2$ for all $n \in \mathbb{N}$. If $\{f_n\}$ did converge uniformly to 0 on $[0, \infty)$, pick $N \in \mathbb{N}$ such that for all $n \geq \mathbb{N}$ and for all $x \in [0, \infty)$,

$$|f_n(x) - f(x)| < \frac{1}{3}.$$

This must be true for $x = 1/N \in [0, \infty)$ and $n = N$ in particular. This demands $|f_N(1/N) - f(1/N)| = 1/2 < 1/3$, which is absurd.

We show that $f_n \to 1$ uniformly on $[a, \infty)$ where $a > 0$. Since $x \geq a$, we have

$$M_n = \sup |f_n(x) - 1| = \sup \frac{|nx - (1 + nx)|}{1 + nx} = \sup \frac{1}{1 + nx} \leq \frac{1}{1 + na} < \frac{1}{na} \to 0.$$

Thus, $M_n \to 0$ which means that $f_n \to 1$ uniformly on $[a, \infty)$.

Formally, given $\epsilon > 0$, pick $N \in \mathbb{N}$ such that $Na\epsilon > 1$. Thus, for all $n \geq N$ and $x \in [a, \infty)$,

$$|f_n(x) - 1| = \left| \frac{nx}{1+nx} - 1 \right| = \frac{1}{1+nx} < \frac{1}{nx} \leq \frac{1}{Na} < \epsilon.$$

This shows that $f_n \to 1$ uniformly on $[a, \infty)$.

(iv) We have been given the functions

$$f_n : [0, \infty) \to \mathbb{R}, \qquad f_n(x) = \frac{x^n}{1+x^n}.$$

We first note that $x^n \to 0$ when $x \in [0, 1)$ and $x^n \to 1$ when $x = 1$. The latter follows trivially. The former also follows trivially when $x = 0$. Let $t = x^{-1} - 1 > 0$ when $0 < x < 1$, since $x^{-n} > 1$. Rearranging, $1/x = 1 + t$, so

$$x^{-n} = (1+t)^n = 1 + nt + \cdots + t^n \geq nt.$$

This means that $x^n \leq 1/nt \to 0$, which gives $x^n \to 0$ as desired.

Now, when $0 \leq x < 1$, we have $x^n \to 0$ so $f_n(x) \to 0$. When $x = 1$, we have $f_n(x) = 1/2$ so $f_n(1) \to 1/2$. When $x > 1$, we have $f_n(x) = 1/(x^{-n} + 1) \to 1$ because $x^{-n} = (1/x)^n \to 0$ since $1/x < 1$. Thus, we set

$$f : [0, \infty) \to \mathbb{R}, \qquad f(x) = \begin{cases} 0, & \text{if } x < 1 \\ \frac{1}{2} & \text{if } x = 1 \\ 1 & \text{if } x > 1 \end{cases},$$

and write $f_n \to f$ pointwise on $[0, \infty)$. By restricting the domain, we also say that $f_n \to f$ on $[0, 1]$ and $f_n \to 0$ on $[0, a)$ where $0 < a < 1$.

Now, the limit function $f$ is discontinuous at $x = 1$, yet the functions $f_n$ are all continuous since they are ratios of polynomial functions with non-zero denominators. Thus, $\{f_n\}$ does not converge uniformly on $[0, \infty)$, nor on $[0, 1]$.

Another way to see this is to see this is that if $f_n \to f$ uniformly on $[0, 1]$, then we can pick $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in [0, \infty)$,

$$|f_n(x) - f(x)| < \frac{1}{4}.$$

This must hold in particular for $x = (1/2)^{1/N} < 1$ and $n = N$, hence

$$\left| \frac{1/2}{1 + 1/2} - 0 \right| = \frac{1}{3} < \frac{1}{4},$$

which is absurd. Thus, $\{f_n\}$ doesn't converge uniformly on $[0, 1]$, nor on $[0, \infty)$.

When $0 < a < 1$, we show that $f_n \to 0$ on $[0, a)$ uniformly. Note that when $0 < x < a < 1$, we have $x^n < a^n$, and when $x = 0$, we see $f_n(x) = 0$, so

$$M_n = \sup |f_n(x) - 0| = \sup \frac{x^n}{1+x^n} \leq \sup x^n \leq a^n \to 0.$$

The last limit follows since $a \in [0, 1)$. Thus, $M_n \to 0$, which means that $f_n \to 0$ uniformly on $(0, a)$ where $0 < a < 1$.

Formally, given $\epsilon > 0$, we choose $N \in \mathbb{N}$ such that $Nt\epsilon > 1$, where $t = a^{-1} - 1$. We have already shown that $a^n \leq 1/nt$. Now, for all $n \geq N$ and $x \in (0, a)$, we have

$$|f_n(x) - 0| = \frac{x^n}{1+x^n} < x^n < a^n < \frac{1}{nt} \leq \frac{1}{Nt} < \epsilon,$$

and $f_n(0) = 0 < \epsilon$. Thus, $f_n \to f$ uniformly on $[0, a)$.

(v) We have been given the functions

$$f_n\colon [0,\infty) \to \mathbb{R}, \qquad f_n(x) = \frac{\sin nx}{1+nx}.$$

When $x > 0$, we see that $\{\sin nx\}$ is a bounded sequence, while $1/(1+nx) < 1/nx \to 0$. This gives $f_n(x) \to 0$. Also, $f_n(0) = 0$, so $f_n(0) \to 0$. Thus, $f_n \to 0$ pointwise on $[0,\infty)$. By restricting the domain, we see that $f_n \to 0$ on $[a,\infty)$, where $a > 0$.

We show that $\{f_n\}$ does not converge uniformly on $[0,\infty)$. Note that $f_n(1/n) = \sin(1)/2 > 0.4$, which means

$$M_n = \sup |f_n(x) - 0| \geq |f_n(1/n)| > 0.4$$

for all $n \in \mathbb{N}$. Thus, $M_n$ does not converge to 0, so $\{f_n\}$ cannot converge uniformly on $[0,\infty)$.

Another way to see this is that if $f_n \to 0$ uniformly on $[0,\infty)$, then we could choose $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in [0,\infty)$,

$$|f_n(x) - 0| < 0.4.$$

This should hold for $x = 1/N \in [0,\infty)$ and $n = N$ in particular. This demands $|f_N(1/N)| = \sin(1)/2 < 0.4$, which is absurd.

We show that $f_n \to 0$ uniformly on $[a,\infty)$, where $a > 0$. Since $x \geq a$ and $|\sin x| \leq 1$, we can write

$$M_n = \sup |f_n(x) - 0| = \sup \frac{|\sin nx|}{1+nx} \leq \frac{1}{1+na} \to 0.$$

This means that $M_n \to 0$, so $f_n \to 0$ uniformly on $[a,\infty)$ where $a > 0$.

Formally, given $\epsilon > 0$, we choose $N \in \mathbb{N}$ such that $Na\epsilon > 1$. Thus, for all $n \geq N$ and $x \in [a,\infty)$, we have

$$|f_n(x) - 0| = \frac{|\sin nx|}{1+nx} \leq \frac{1}{1+nx} \leq \frac{1}{na} \leq \frac{1}{Na} < \epsilon.$$

This means that $f_n \to 0$ uniformly on $[a,\infty)$.

(vi) We have been given the functions

$$f_n\colon \mathbb{R} \to \mathbb{R}, \qquad f_n(x) = e^{-nx}.$$

Note that whenever $x < 0$, we have $e^{-x} > 1$ because

$$e^y = 1 + y + \frac{y^2}{2} + \cdots > 1$$

when $-x = y > 0$. Thus, $(e^{-x})^n = e^{-nx} \to \infty$, so $\{f_n\}$ does not converge on the entirety of $\mathbb{R}$. When $x = 0$, $e^{-n0} = 1$ so $f_n(0) \to 1$. When $x > 0$, $e^{-x} = 1/e^x < 1$, so $f_n(x) = e^{-nx} \to 0$. Thus, setting

$$f\colon [0,\infty) \to \mathbb{R}, \qquad f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x > 0 \end{cases},$$

we have $f_n \to f$ pointwise on $[0,\infty)$.

Note that $\{f_n\}$ does not converge uniformly to $f$ on $[0,\infty)$ since each $f_n$ is continuous, but the limit function $f$ is discontinuous at 0.

Alternatively, examine $f_n(1/n) = e^{-1} > 0$, which means that

$$M_n = \sup |f_n(x) - f(x)| \geq |f_n(1/n) - f(1/n)| = e^{-1},$$

so $M_n$ cannot converge to 0.

Formally, if $f_n \to f$ uniformly on $[0, \infty)$, then we could choose $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in [0, \infty)$,

$$|f_n(x) - f(x)| < e^{-1}.$$

This should hold in particular for $x = 1/N$ and $n = N$. This demands

$$|f_N(1/N) - f(1/N)| = e^{-1} < e^{-1},$$

which is absurd.

(vii) We have been given the functions

$$f_n \colon \mathbb{R} \to \mathbb{R}, \qquad f_n(x) = x^2 e^{-nx}.$$

We have already seen that $e^{-nx} \to \infty$ when $x < 0$, $e^{-nx} \to 0$ when $x > 0$. Thus, $x^2 e^{-nx} \to \infty$ when $x < 0$ and $x^2 e^{-nx} \to 0$ when $x \geq 0$. This means that $\{f_n\}$ does not converge on $\mathbb{R}$, but $f_n \to 0$ pointwise on $[0, \infty)$.

We show that $f_n \to 0$ uniformly on $[0, \infty)$. Note that $f_n(0) = 0$ and when $x > 0$, we have

$$e^{nx} = 1 + nx + \frac{1}{2}n^2x^2 + \cdots > \frac{1}{2}n^2x^2,$$

so $e^{-nx} < 2/n^2x^2$. Thus, $f_n(x) = x^2e^{-nx} < 2/n^2$, so

$$M_n = \sup |f_n(x) - 0| = \sup x^2 e^{-nx} \leq \frac{2}{n^2} \to 0.$$

This means that $M_n \to 0$, so $f_n \to 0$ uniformly on $[0, \infty)$.

Formally, given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $N^2\epsilon > 2$. Thus, for all $n \geq N$ and $x \in [0, \infty)$, we have

$$|f_n(x) - 0| = x^2 e^{-nx} < \frac{2}{n^2} \leq \frac{2}{N^2} < \epsilon.$$

This shows that $f_n \to 0$ uniformly on $[0, \infty)$.

(viii) We have been given the functions

$$f_n \colon \mathbb{R} \to \mathbb{R}, \qquad f_n(x) = n^2x^2e^{-nx}.$$

Again, recall that when $x < 0$, we have $x^2e^{-nx} \to \infty$, so $f_n(x) = n^2x^2e^{-nx} \to \infty$. This means that $\{f_n\}$ does not converge on $\mathbb{R}$. When $x = 0$, we have $f_n(0) = 0$ and when $x > 0$, we see that

$$e^{nx} = 1 + nx + \frac{1}{2}n^2x^2 + \frac{1}{6}n^3x^3 + \cdots > \frac{1}{6}n^3x^3,$$

so $f_n(x) = n^2x^2e^{-nx} < n^2x^2 \cdot 6/n^3x^3 = 6/nx \to 0$. Also, $f_n(x) \geq 0$, so $f_n \to 0$ pointwise on $[0, \infty)$.

We show that $\{f_n\}$ does not converge uniformly on $[0, \infty)$ by noting that $f_n(1/n) = e^{-1} > 0$, so

$$M_n = \sup |f_n(x) - 0| \geq |f_n(1/n) - 0| = e^{-1}.$$

This means that $M_n$ cannot converge to 0, so $\{f_n\}$ cannot converge uniformly on $[0, \infty)$.

Alternatively, if $f_n \to 0$ uniformly on $[0, \infty)$, we could choose $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in [0, \infty)$, we have

$$|f_n(x) - 0| < e^{-1}.$$

This should hold in particular for $x = 1/N$ and $n = N$. This demands

$$|f_N(1/N) - 0| = e^{-1} < e^{-1},$$

which is absurd.

(ix) We have been given the functions

$$f_n\colon [0,2] \to \mathbb{R}, \qquad f_n(x) = \frac{x^n}{1+x^n}.$$

Recall that we've already shown that $f_n \to f$ pointwise where

$$f\colon [0,2] \to \mathbb{R}, \qquad f(x) = \begin{cases} 0, & \text{if } x < 1 \\ \frac{1}{2} & \text{if } x = 1 \\ 1 & \text{if } x > 1 \end{cases},$$

Again, this convergence is not uniform since each $f_n$ is continuous on $[0,2]$, but the limiting function $f$ is discontinuous at 1.

(x) We have been given the functions

$$f_n\colon [0,1] \to \mathbb{R}, \qquad f_n(x) = \frac{1}{(1+x)^n}.$$

Recall that we've shown that when $0 < x < 1$, we have $x^n \to 0$. Thus, when $0 < x \le 1$, we have $1 < 1+x \le 2$ hence $1/2 \le 1/(1+x) < 1$, so $1/(1+x)^n \to 0$. When $x = 0$, we have $1/(1+x)^n = 1$. Thus, setting

$$f\colon [0,1] \to \mathbb{R}, \qquad f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x > 0 \end{cases},$$

we have $f_n \to f$ pointwise on $[0,1]$.

We see that $\{f_n\}$ does not converge uniformly on $[0,1]$ because each of the functions $f_n$ is continuous, being the ratio of polynomials with a non-zero denominator, but the limiting function $f$ is discontinuous at 0.

Alternatively, if $f_n \to f$ uniformly on $[0,1]$, then we could choose $N \in \mathbb{N}$ such that for all $n \ge N$ and $x \in [0,1]$, we have

$$|f_n(x) - f(x)| < \frac{1}{3}.$$

This should hold in particular for $x = 2^{1/N} - 1 \in [0,1]$ and $n = N$. Since $x > 0$, this demands

$$f_N(2^{1/N} - 1) = \frac{1}{2} < \frac{1}{3},$$

which is absurd.

**Solution 2.** We have been given the sequences of functions $\{f_n\}$ and $\{g_n\}$ such that $f_n \to f$ and $g_n \to g$ uniformly on some domain $X$. We claim that $f_n + g_n \to f + g$ uniformly on $X$.

To show this, let $\epsilon > 0$ be arbitrary. From the uniform convergence of $f_n \to f$ and $g_n \to g$, we choose $N_1, N_2 \in \mathbb{N}$ such that for all $x \in X$,

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}$$

whenever $n \ge N_1$ and

$$|g_n(x) - g(x)| < \frac{\epsilon}{2}$$

whenever $n \ge N_2$. Thus, whenever $x \in X$ and $n \ge N = N_1 + N_2$, we have

$$\begin{aligned} |(f_n(x) + g_n(x)) - (f(x) + g(x))| &= |(f_n(x) - f(x)) + (g_n(x) - g(x))| \\ &\le |f_n(x) - f(x)| + |g_n(x) - g(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

The second line follows by the triangle inequality. Thus, $f_n + g_n \to f + g$ uniformly, as desired.

**Solution 3.** We have been given the functions

$$f_n \colon \mathbb{R} \to \mathbb{R}, \qquad f_n(x) = x + \frac{1}{n}.$$

We first show that $f_n \to f$ where $f \colon \mathbb{R} \to \mathbb{R}$, $f(x) = x$ is the identity function. Moreover, this convergence is uniform because

$$M_n = \sup |f_n(x) - f(x)| = \sup \left| x + \frac{1}{n} - x \right| = \frac{1}{n} \to 0.$$

The last limit follows from the Archimedean property of the reals: given any $\epsilon > 0$, we can choose $N \in \mathbb{N}$ such that $N\epsilon > 1$, so $1/n < \epsilon$ for all $n \geq N$. Thus, $M_n \to 0$ so $f_n \to f$ uniformly on $\mathbb{R}$.

Now, we examine $\{f^2\}$. Note that

$$f_n^2(x) = x^2 + \frac{2x}{n} + \frac{1}{n^2},$$

and we have the limits $x^2 \to x^2$, $2x/n \to 0$ and $1/n^2 \to 0$. The last two follow from $1/n \to 0$, since $x \in \mathbb{R}$ is fixed. Thus, $f_n^2 \to f^2$ pointwise, where $f^2(x) = x^2$. Now,

$$f_n^2(x) - f^2(x) = \frac{2x}{n} + \frac{1}{n^2} > \frac{2x}{n}.$$

This means that $f_n^2(n) - f^2(n) > 2$, so

$$M_n = \sup |f_n^2(x) - f^2(x)| \geq |f_n^2(n) - f^2(n)| > 2.$$

Thus, $M_n$ cannot converge to 0, so $\{f_n^2\}$ cannot converge uniformly on $\mathbb{R}$.