MA 2102 : Mathematical Methods II

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Fourier Series and Transforms (M.L. Boas, Chapter 7)

Section 6. Problem 14 Use the Fourier expansion of the following function (as seen in problem 5.7),

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases},$$

to show that $S = \sum_{\text{odd } n} 1/n^2 = \pi^2/8$. Try $x = 0, \pi, \pi/2$.

Solution. We recall that the Fourier expansion of f is given by

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[-\frac{1}{n^2\pi}(1 - \cos n\pi) \cos nx - \frac{1}{n} \cos n\pi \sin nx \right].$$

Now, note that f satisfies the Dirichlet conditions, since it is a single-valued periodic function of period 2π, defined between $-\pi$ and $+\pi$, has a finite number of extrema, with a finite number of discontinuities (at 0 and π). Also,

$$\int_{-\pi}^{+\pi} |f(x)|\, dx = \int_0^{\pi} x\, dx = \frac{\pi^2}{2},$$

which is finite. Hence, our Fourier series does indeed converge to f. Furthermore, the series at the discontinuities 0 and π converges to the average of the left and right hand limits, i.e. to the midpoints of the jump discontinuities. Thus,

$$f(0) = \frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x) \right] = \frac{1}{2}[0 + 0] = 0.$$

$$f(\pi) = \frac{1}{2}\left[\lim_{x\to \pi^-} f(x) + \lim_{x\to \pi^+} f(x) \right] = \frac{1}{2}[\pi + 0] = \frac{\pi}{2}.$$

Thus, we first set $x = 0$. Then, all sine terms vanish and all the cosines become unity. Note that $1 - \cos n\pi$ vanishes for even n and becomes 2 for odd n, so we obtain

$$f(0) = 0 = \frac{\pi}{4} - \frac{2}{\pi} \sum_{\text{odd } n} \frac{1}{n^2}.$$

Rearranging, $S = (\pi/4)(\pi/2) = \pi^2/8$.

We may also try $x = \pi$, in which case the sines vanish again, and the cosines follow $\cos n\pi = (-1)^n$. Thus, $\sum_{\text{odd } n} (-1)^n/n^2 = -S$, so we have

$$f(\pi) = \frac{\pi}{2} = \frac{\pi}{4} + \frac{2}{\pi} \sum_{\text{odd } n} \frac{1}{n^2}.$$

Rearranging, we again obtain $S = (\pi/4)(\pi/2) = \pi^2/8$.

Finally, when $x = \pi/2$, the cosines vanish and the sines follow $\sin n\pi/2 = 1, 0, -1, 0, \ldots$. Thus, we get an alternating sum

$$f(\pi/2) = \frac{\pi}{2} = \frac{\pi}{4} - \sum_{\text{odd } n} (-1)^n \frac{(-1)^{(n+1)/2}}{n} = \frac{\pi}{4} + 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Thus, we obtain

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

As a side note, from $\sum_{\text{odd } n} 1/n^2 = \pi^2/8$, we see that

$$\sum_{\text{even } n} \frac{1}{n^2} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4}\sum_{\text{even } n} \frac{1}{n^2} + \frac{1}{4}\sum_{\text{odd } n} \frac{1}{n^2} = \frac{1}{4}\sum_{\text{even } n} \frac{1}{n^2} + \frac{\pi^2}{32}.$$

Rearranging, $\sum_{\text{even } n} 1/n^2 = (4/3)(\pi^2/32) = \pi^2/24$. Adding on the odd terms, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

The computation of this particular infinite sum is famously known as the Basel problem.

Section 7. Problem 2. Expand the following periodic function as a Fourier series.

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi/2, \\ 0, & \pi/2 < x < \pi. \end{cases}$$

Solution. We write

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx}.$$

The coefficients for $n \neq 0$ are calculated as

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x) e^{-inx}\, dx = \frac{1}{2\pi i n}\left[1 - e^{-in\pi/2}\right].$$

When $n = 0$,

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\, dx = \frac{1}{2\pi}\cdot\frac{\pi}{2} = \frac{1}{4}.$$

Thus,

$$f(x) = \frac{1}{4} + \frac{1}{2\pi i}\sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{1}{n}\left[1 - e^{-in\pi/2}\right] e^{inx}.$$

We can rewrite this as

$$\begin{aligned} f(x) &= \frac{1}{4} + \frac{1}{2\pi i}\sum_{n=1}^{\infty} \frac{1}{n}\left[1 - e^{-in\pi/2}\right] e^{inx} - \frac{1}{n}\left[1 - e^{in\pi/2}\right] e^{-inx} \\ &= \frac{1}{4} + \frac{1}{2\pi i}\sum_{n=1}^{\infty} \frac{1}{n}\left[e^{inx} - e^{-inx}\right] - \frac{1}{n}\left[e^{in(x-\pi/2)} - e^{-in(x-\pi/2)}\right] \\ &= \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{n\pi}\sin nx - \frac{1}{n\pi}\sin(nx - n\pi/2)\right] \\ &= \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{n\pi}\sin nx - \frac{1}{n\pi}\cos\frac{n\pi}{2}\sin nx + \frac{1}{n\pi}\sin\frac{n\pi}{2}\cos nx\right] \\ &= \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{n\pi}\sin\frac{n\pi}{2}\cos nx + \frac{1}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)\sin nx\right]. \end{aligned}$$

This is precisely what we obtained earlier in (5.2).

Problem 7. Expand the following periodic function as a Fourier series.

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$$

Solution. We write

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx}.$$

The coefficients for $n \neq 0$ are calculated as

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)e^{-inx}\,dx = -\frac{1}{2\pi in}\pi e^{-in\pi} + \frac{1}{2\pi in}\int_0^{\pi} e^{-inx}\,dx = -\frac{1}{2in}e^{-in\pi} + \frac{1}{2\pi n^2}\left[e^{-in\pi} - 1\right].$$

When $n = 0$,

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\,dx = \frac{1}{2\pi}\cdot\frac{\pi^2}{2} = \frac{\pi}{4}.$$

Thus,

$$f(x) = \frac{\pi}{4} - \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty}\left[\frac{1}{2in}e^{-in\pi} - \frac{1}{2\pi n^2}e^{-in\pi} + \frac{1}{2\pi n^2}\right]e^{inx}.$$

We can rewrite this as

$$\begin{aligned} f(x) &= \frac{\pi}{4} - \sum_{n=1}^{\infty}\frac{1}{2in}\left[e^{in(x-\pi)} - e^{-in(x-\pi)}\right] - \frac{1}{2\pi n^2}\left[e^{in(x-\pi)} + e^{-in(x-\pi)}\right] + \frac{1}{2\pi n^2}\left[e^{inx} - e^{-inx}\right] \\ &= \frac{\pi}{4} - \sum_{n=1}^{\infty}\frac{1}{n}\sin(nx - n\pi) - \frac{1}{\pi n^2}\cos(nx - n\pi) + \frac{1}{\pi n^2}\cos nx \\ &= \frac{\pi}{4} - \sum_{n=1}^{\infty}\frac{1}{n}\cos n\pi\sin nx - \frac{1}{\pi n^2}\cos n\pi\cos nx + \frac{1}{\pi n^2}\cos nx \\ &= \frac{\pi}{4} - \sum_{n=1}^{\infty}\left[\frac{1}{\pi n^2}(1 - \cos n\pi)\cos nx + \frac{1}{n}\cos n\pi\sin nx\right]. \end{aligned}$$

This is precisely what we obtained earlier in (5.7).

Problem 11. Expand the following periodic function as a Fourier series.

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ \sin x, & 0 < x < \pi. \end{cases}$$

Solution. We write

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx}.$$

The coefficients for $n \neq 0, \pm 1$ are calculated as

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)e^{-inx}\,dx = \frac{1}{4\pi i}\int_0^{\pi}(e^{ix} - e^{-ix})e^{-inx}\,dx \quad = -\frac{1}{4\pi}\left[\frac{e^{i\pi(1-n)} - 1}{1-n} + \frac{e^{-i\pi(1+n)} - 1}{1+n}\right].$$

For odd n, note that $c_n = 0$. Thus,

$$c_{2n} = \frac{1}{2\pi}\left[\frac{1}{1-2n} + \frac{1}{1+2n}\right] = -\frac{1}{\pi}\cdot\frac{1}{4n^2-1}.$$

When $n = 0$,

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\,dx = \frac{1}{2\pi}\cdot 2 = \frac{1}{\pi}.$$

When $n = 1$,

$$c_1 = \frac{1}{4\pi i}\int_0^\pi (e^{ix} - e^{-ix})e^{-ix}\,dx = \frac{1}{4\pi i}\left[\pi - \frac{1}{2i}e^{-2\pi i} + \frac{1}{2i}\right] = \frac{1}{4i}.$$

When $n = -1$,

$$c_{-1} = \frac{1}{4\pi i}\int_0^\pi (e^{ix} - e^{-ix})e^{ix}\,dx = \frac{1}{4\pi i}\left[-\pi + \frac{1}{2i}e^{-2\pi i} - \frac{1}{2i}\right] = -\frac{1}{4i}.$$

Thus,

$$f(x) = \frac{1}{\pi} + \frac{1}{4i}e^{ix} - \frac{1}{4i}e^{-ix} - \frac{1}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{1}{4n^2-1}e^{2inx}.$$

We can rewrite this as

$$\begin{aligned} f(x) &= \frac{1}{\pi} + \frac{1}{2}\cdot\frac{1}{2i}(e^{ix} - e^{-ix}) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2}\cdot\frac{1}{4n^2-1}(e^{2inx} + e^{-2inx}) \\ &= \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx. \end{aligned}$$

This is precisely what we obtained earlier in (5.11).

Section 8. Problem 7. Expand the following periodic function as a Fourier series.

$$f(x) = \begin{cases} 0, & -\ell < x < 0, \\ x, & 0 < x < \ell. \end{cases}$$

Solution. We write

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx/\ell} = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}.$$

The coefficients c_n for $n \neq 0$ are calculated as

$$c_n = \frac{1}{2\ell}\int_{-\ell}^{+\ell} f(x)e^{-in\pi x/\ell}dx = -\frac{1}{2\pi in}\ell e^{-in\pi} + \frac{1}{2\pi in}\int_0^\ell e^{-in\pi x/\ell}dx = -\frac{\ell}{2\pi in}e^{-in\pi} + \frac{\ell}{2\pi^2 n^2}\left[e^{-in\pi} - 1\right].$$

When $n = 0$,

$$c_0 = a_0 = \frac{1}{2\ell}\int_{-\ell}^{+\ell} f(x)\,dx = \frac{1}{2\ell}\cdot\frac{\ell^2}{2} = \frac{\ell}{4}.$$

Thus,

$$f(x) = \frac{\ell}{4} - \frac{\ell}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty}\left[\frac{1}{2in}e^{-in\pi} - \frac{1}{2\pi n^2}e^{-in\pi} + \frac{1}{2\pi n^2}\right]e^{in\pi x/\ell}.$$

The coefficients a_n and b_n are calculated for $n > 0$ as,

$$a_n = \frac{1}{\ell}\int_{-\ell}^{+\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx = \frac{1}{\ell}\int_0^\ell x\cos\frac{n\pi x}{\ell}\,dx = \frac{1}{n\pi}x\sin\frac{n\pi x}{\ell}\Big|_0^\ell - \frac{1}{n\pi}\int_0^\ell \sin\frac{n\pi x}{\ell}\,dx = -\frac{\ell}{n^2\pi^2}(1 - \cos n\pi),$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{+\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx = \frac{1}{\ell}\int_0^\ell x\sin\frac{n\pi x}{\ell}\,dx = -\frac{1}{n\pi}x\cos\frac{n\pi x}{\ell}\Big|_0^\ell + \frac{1}{n\pi}\int_0^\ell \cos\frac{n\pi x}{\ell}\,dx = -\frac{\ell}{n\pi}\cos n\pi,$$

Thus,

$$f(x) = \frac{\ell}{4} - \frac{\ell}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi}(1 - \cos n\pi)\cos\frac{n\pi x}{\ell} + \frac{1}{n}\cos n\pi\sin\frac{n\pi x}{\ell}\right].$$

Note that our new solutions are precisely the old ones, scaled by ℓ/π and with the substitution $x \mapsto \pi x/\ell$. This is because our new function is merely the old one scaled by a factor of ℓ/π along both axes.