MA 2102 : Mathematical Methods II

Satvik Saha, 19MS154 September 24, 2020

Fourier Series and Transforms (M.L. Boas, Chapter 7)

Section 3. Problem 5. Using the definition of a periodic function, show that a sum of terms corresponding to a fundamental musical tone and its overtones has the period of the fundamental.

Solution. Let the fundamental tone be the function $f\colon \mathbb{R} \to \mathbb{R}$. The harmonics are simply $f_n(t) = f(nt)$ for non-zero integers n. Suppose that f has a period p, so for all $t \in \mathbb{R}$,

$$f(t) = f(t + p).$$

We first prove that $f(t) = f(t + kp)$ for all $k \in \mathbb{N}$. The base case of $k = 1$ follows directly from the definition. Suppose that this holds for some $k' \in \mathbb{N}$, i.e. $f(t + k'p) = f(t)$. Then,

$$f(t + (k' + 1)p) = f((t + k'p) + p) = f(t + k'p) = f(t).$$

Thus, the desired statement is true by induction. We also note that $f(t - kp) = f((t - kp) + kp) = f(t)$, so our statement is also true for all negative integers k. Now, note that

$$f_n(t + p) = f(n(t + p)) = f(nt + np) = f(nt) = f_n(t).$$

Thus, by definition, the harmonics f_n are also periodic with period p, the same as the fundamental. Now consider an arbitrary linear combination of harmonics,

$$g = \lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_m f_m,$$

where m is the highest harmonic present and λ_i are real scalars. Thus,

$$g(t + p) = \sum_{n=1}^{m} \lambda_n f_n(t + p) = \sum_{n=1}^{m} \lambda_n f_n(t) = g(t).$$

Thus, g is periodic with period p, the same as the fundamental tone.
We further consider an infinite sum,

$$g = \sum_{n=1}^{\infty} \lambda_n f_n.$$

If g is well defined on its domain, i.e. the sum always converges, we can apply exactly the same argument as before 'in the limit $m \to \infty$' (essentially taking a limit of partial sums g_m, all of which satisfy $g_m(t) = g_m(t + p)$) to conclude that g still has a period p.

Section 4. Problem 13. Show that

$$\int_a^b \sin^2 kx\, dx = \int_a^b \cos^2 kx\, dx = \frac{1}{2}(b - a),$$

if $k(b - a)$ is an integral multiple of π, or if kb and ka are both integral multiples of $\pi/2$.

Solution. For the first case, suppose $k(b - a) = n\pi$ for some intger n. If $n = 0$, either $b - a = 0$ in which case the equality is trivial, or $k = 0$, in which case it is false. Otherwise, we use the substitution $nu = kx$ to write

$$I = \int_a^b \sin^2 kx\, dx = \frac{n}{k} \int_{a\pi/(b-a)}^{b\pi/(b-a)} \sin^2 nu\, du, \qquad J = \int_a^b \cos^2 kx\, dx = \frac{n}{k} \int_{a\pi/(b-a)}^{b\pi/(b-a)} \cos^2 nu\, du.$$

Note that $\sin^2 nx$ and $\cos^2 nx$ have periods of π, because $\sin(nx+n\pi) = (-1)^n \sin x$ and $\cos(nx+n\pi) = (-1)^n \cos x$. Also, $\sin^2(nx) = \sin^2(-nx)$ and $\cos^2(nx) = \cos^2(-nx)$, so

$$\int_0^\pi \sin^2 nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin^2 nx\,dx = \frac{\pi}{2}, \qquad \int_0^\pi \cos^2 nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos^2 nx\,dx = \frac{\pi}{2}.$$

Note that for any periodic function f with period β,

$$\begin{aligned}\int_\alpha^{\alpha+\beta} f(t)\,dt &= \int_0^\beta f(t)\,dt + \int_\beta^{\alpha+\beta} f(t)\,dt - \int_0^\alpha f(t)\,dt \\ &= \int_0^\beta f(t)\,dt + \int_{\beta-\beta}^{\alpha+\beta-\beta} f(t+\beta)\,dt - \int_0^\alpha f(t)\,dt \\ &= \int_0^\beta f(t)\,dt + \int_0^\alpha f(t)\,dt - \int_0^\alpha f(t)\,dt \\ &= \int_0^\beta f(t)\,dt.\end{aligned}$$

Thus, for any α,

$$\int_\alpha^{\alpha+\pi} \sin^2 nu\,du = \int_\alpha^{\alpha+\pi} \cos^2 nu\,du = \frac{\pi}{2}.$$

Specifically, we set $\alpha = a\pi/(b-a)$, noting that $a\pi/(b-a) + \pi = b\pi/(b-a)$, so it follows that

$$I = J = \frac{n\pi}{2k} = \frac{1}{2}(b-a).$$

□

Now consider the case $ka = m\pi/2$, $kb = n\pi/2$ for integers m, n. If $m = n$, the integrals are trivially zero. Otherwise, we use the substitution $u = kx$ to write

$$I = \int_a^b \sin^2 kx\,dx = \frac{1}{k}\int_{m\pi/2}^{n\pi/2} \sin^2 u\,du, \qquad J = \int_a^b \cos^2 kx\,dx = \frac{1}{k}\int_{m\pi/2}^{n\pi/2} \cos^2 u\,du.$$

Suppose $n > m$. Thus, our integral splits into the sums

$$I = \frac{1}{k}\sum_{\ell=m}^{n-1}\int_{\ell\pi/2}^{(\ell+1)\pi/2} \sin^2 u\,du, \qquad J = \frac{1}{k}\sum_{\ell=m}^{n-1}\int_{\ell\pi/2}^{(\ell+1)\pi/2} \cos^2 u\,du.$$

Again, if $n < m$, we write

$$I = -\frac{1}{k}\sum_{\ell=n}^{m-1}\int_{\ell\pi/2}^{(\ell+1)\pi/2} \sin^2 u\,du, \qquad J = -\frac{1}{k}\sum_{\ell=n}^{m-1}\int_{\ell\pi/2}^{(\ell+1)\pi/2} \cos^2 u\,du.$$

In either case, we have a sum of $|n-m|$ terms.

Now, note that the substitution $u = \pi/2 - x$ yields

$$\int_0^{\pi/2} \sin^2 x\,dx = -\int_{\pi/2}^0 \sin^2(\pi/2 - u)\,du = \int_0^{\pi/2} \cos^2 x\,dx.$$

Thus,

$$\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \cos^2 x\,dx = \frac{1}{2}\int_0^{\pi/2} \sin^2 x + \cos^2 x\,dx = \frac{1}{2}\int_0^{\pi/2} dx = \frac{\pi}{4}.$$

Also, since $\sin^2(x + n\pi/2) = \sin^2 x$, $\cos^2(x+n\pi/2) = \cos^2 x$ for even n and $\sin^2(x+n\pi/2) = \cos^2 x$, $\cos^2(x+n\pi/2) = \sin^2 x$ for odd n, we have

$$\int_{\ell\pi/2}^{(\ell+1)\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \sin^2(x + \ell\pi/2)\,dx = \frac{\pi}{4},$$

$$\int_{\ell\pi/2}^{(\ell+1)\pi/2} \cos^2 x \, dx = \int_0^{\pi/2} \cos^2(x + \ell\pi/2) \, dx = \frac{\pi}{4}.$$

Thus, each term in the sums I and J is simply $\pi/4$, which means that

$$I = J = \frac{(n-m)\pi}{4k} = \frac{1}{2}(b-a). \qquad \square$$

Section 5. Problem 2. Expand the following periodic function as a Fourier series.

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi/2, \\ 0, & \pi/2 < x < \pi. \end{cases}$$

Solution. We write

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

The coefficients are calculated as

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x) \, dx = \frac{1}{2\pi} \cdot \frac{\pi}{2} = \frac{1}{4}.$$

For $n > 0$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_0^{\pi/2} \cos nx \, dx = \frac{1}{n\pi} \sin \frac{n\pi}{2},$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_0^{\pi/2} \sin nx \, dx = \frac{1}{n\pi} \left(1 - \cos \frac{n\pi}{2}\right).$$

Note that a_{2n} vanish. The sequences repeat $n\pi a_n = 1, 0, -1, 0, \ldots$ and $n\pi b_n = 1, 2, 1, 0, \ldots$.

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \left[\frac{1}{n\pi} \sin \frac{n\pi}{2} \cos nx + \frac{1}{n\pi} \left(1 - \cos \frac{n\pi}{2}\right) \sin nx \right].$$

Problem 7. Expand the following periodic function as a Fourier series.

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$$

Solution. We write

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

The coefficients are calculated as

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x) \, dx = \frac{1}{2\pi} \int_0^{\pi} x \, dx = \frac{1}{2\pi} \cdot \frac{\pi^2}{2} = \frac{\pi}{4}.$$

For $n > 0$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_0^{\pi} x \cos nx \, dx = \frac{1}{n\pi} x \sin nx \Big|_0^{\pi} - \frac{1}{n\pi} \int_0^{\pi} \sin nx \, dx = -\frac{1}{n^2\pi}(1 - \cos n\pi),$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_0^{\pi} x \sin nx \, dx = -\frac{1}{n\pi} x \cos nx \Big|_0^{\pi} + \frac{1}{n\pi} \int_0^{\pi} \cos nx \, dx = -\frac{1}{n} \cos n\pi,$$

Note that a_{2n} vanish. The sequences repeat $n^2\pi a_n = -2, 0, -2, 0, \ldots$ and $n\pi b_n = 1, -1, 1, -1, \ldots$.

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[-\frac{1}{n^2\pi}(1 - \cos n\pi) \cos nx - \frac{1}{n} \cos n\pi \sin nx \right].$$

Problem 11. Expand the following periodic function as a Fourier series.

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ \sin x, & 0 < x < \pi. \end{cases}$$

Solution. We write

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

The coefficients are calculated as

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)\, dx = \frac{1}{2\pi} \int_0^{\pi} \sin x\, dx = \frac{1}{2\pi} \cdot 2 = \frac{1}{\pi}.$$

For $n > 0$,

$$\begin{aligned} a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_0^{\pi} \sin x \cos nx\, dx &= \frac{1}{2\pi} \int_0^{\pi} \sin{(n+1)x} - \sin{(n-1)x}\, dx \\ &= \frac{1}{2\pi} \left[-\frac{1}{n+1} \cos{(n+1)\pi} + \frac{1}{n-1} \cos{(n-1)\pi} - \frac{2}{n^2-1} \right] \\ &= \frac{1}{2\pi} \left[-((-1)^n + 1) \frac{2}{n^2-1} \right] \\ &= \begin{cases} 0, & n \text{ odd}, \\ -2/(n^2-1)\pi, & n \text{ even}. \end{cases} \\ b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin nx\, dx = \frac{1}{\pi} \int_0^{\pi} \sin x \sin nx\, dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin x \sin nx\, dx \\ &= \begin{cases} 1/2, & n = 1, \\ 0, & n > 1. \end{cases} \end{aligned}$$

Thus,

$$f(x) = \frac{1}{\pi} + \frac{1}{2} \sin x - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cos 2nx.$$