MA 2102 : Mathematical Methods II

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Partial Differential Equations (M.L. Boas, Chapter 13)

Section 1. Problem 2.

(a) Show that the expression $u = \sin(x - vt)$ describing a sinusoidal wave satisfies the wave equation

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}.$$

Show that in general, $f(x - vt)$ and $f(x + vt)$ satisfy the wave equation, where f is any function with a second derivative.

Solution. Note that in one dimension, the laplacian ∇^2 is simply the operator $\partial^2/\partial x^2$ (the other two spatial derivatives cause $u(x - vt)$ to vanish). Thus, we compute derivatives

$$\frac{\partial}{\partial x}u = \cos(x - vt), \qquad \frac{\partial^2}{\partial x^2}u = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}u\right) = -\sin(x - vt).$$

$$\frac{\partial}{\partial t}u = -v\cos(x - vt), \qquad \frac{\partial^2}{\partial t^2}u = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}u\right) = -v^2\sin(x - vt).$$

Thus, $\ddot{u} = v^2 u''$ as desired, so u does indeed satisfy the wave equation.

For some general, twice differentiable function f, we can do the same. Let f' and f'' denote the first and second derivatives of f respectively. Then,

$$\frac{\partial}{\partial x}f(x \pm vt) = f'(x \pm vt), \qquad \frac{\partial^2}{\partial x^2}f(x \pm vt) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}f(x \pm vt)\right) = f''(x \pm vt).$$

$$\frac{\partial}{\partial t}f(x \pm vt) = \pm v f'(x \pm vt), \qquad \frac{\partial^2}{\partial t^2}f(x \pm vt) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}f(x \pm vt)\right) = v^2 f''(x \pm vt).$$

Clearly, $\ddot{u} = v^2 u''$, where $u = f(x \pm vt)$ (these are actually two separate cases, which we present together for brevity). This means that $f(x \pm vt)$ are both solutions of the wave equation.

(b) Show that

$$u(r, t) = \frac{1}{r}f(r \pm vt)$$

both satisfy the wave equation in spherical coordinates.

Solution. We must verify that

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}u\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}u.$$

The first step follows by the product rule, which simplifies our work.

$$\begin{aligned}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}u\right) - r\frac{\partial^2}{\partial r^2}(ru) &= 2r\frac{\partial}{\partial r}u + r^2\frac{\partial^2}{\partial r^2}u - r\frac{\partial}{\partial r}\left(u + r\frac{\partial}{\partial r}u\right) \\ &= 2r\frac{\partial}{\partial r}u + r^2\frac{\partial^2}{\partial r^2}u - 2r\frac{\partial}{\partial r}u - r^2\frac{\partial^2}{\partial r^2}u \\ &= 0.\end{aligned}$$

Thus, we need only calculate

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(ru) = \frac{1}{r}\frac{\partial^2}{\partial r^2}f(r \pm vt) = \frac{1}{r}f''(r \pm vt), \qquad \frac{\partial^2}{\partial t^2}u = \frac{1}{r}\frac{\partial^2}{\partial t^2}f(r \pm vt) = \frac{v^2}{r}f''(r \pm vt).$$

Thus, $\ddot{u} = v^2 u''$, so $f(r \pm vt)/r$ are indeed solutions of the wave equation.

Section 4. Problem 2. A string of length ℓ has a zero initial velocity and a displacement $y_0(x)$, described as

$$y_0(x) = \begin{cases} 4hx/\ell, & \text{if } 0 \le x \le \ell/4, \\ 2h - 4hx/\ell, & \text{if } \ell/4 < x \le \ell/2, \\ 0, & \text{if } \ell/2 < x \le \ell. \end{cases}$$

Find the displacement as a function of x and t.

Solution. We seek a solution $y(x,t)$ to the wave equation

$$\frac{\partial^2}{\partial x^2} y = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} y.$$

We perform the separation of variable $y(x,t) = X(x)T(t)$, thus obtaining

$$\frac{1}{X} \frac{d^2}{dx^2} X = \frac{1}{v^2 T} \frac{d^2}{dt^2} T = \text{constant} = -k^2.$$

Note that we chose $-k^2$ to ensure that our solutions do not diverge. Setting $kv = \omega$, we see that this is equivalent to the ODES

$$X'' + k^2 X = 0, \qquad T'' + \omega^2 T = 0.$$

Thus, our solutions look like $X(x) = A\cos kx + B\sin kx$, and $T(t) = C\cos\omega t + D\sin\omega t$. From the boundary conditions $y(0,t) = y(\ell,t) = 0$, we see that $A = 0$ and $k_n = n\pi/\ell$. From the initial condition $v(x) = y_0'(x) = 0$, we see that $T'(0) = \omega D = 0$. Thus, our solution is of the form

$$y(x,t) \;=\; \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{\ell} \cos\frac{n\pi v t}{\ell}.$$

The coefficients A_n are obtained by observing that at $t = 0$, we can write $y_0(x)$ as a sine series. Thus,

$$\begin{aligned} A_n &= \frac{2}{\ell} \int_0^{\ell} y_0(x) \sin\frac{n\pi x}{\ell}\, dx \\ &= \frac{2}{\ell} \int_0^{\ell/4} \frac{4hx}{\ell} \sin\frac{n\pi x}{\ell}\, dx + \frac{2}{\ell} \int_{\ell/4}^{\ell/2} \left(2h - \frac{4hx}{\ell}\right) \sin\frac{n\pi x}{\ell}\, dx \\ &= \frac{2h}{n^2\pi^2} \left[4\sin\frac{n\pi}{4} - n\pi\cos\frac{n\pi}{4} + 4\sin\frac{n\pi}{4} - 4\sin\frac{n\pi}{2} + n\pi\cos\frac{n\pi}{4}\right] \\ &= \frac{8h}{n^2\pi^2} \left[2\sin\frac{n\pi}{4} - \sin\frac{n\pi}{2}\right]. \end{aligned}$$

Thus, we have our solution

$$y(x,t) \;=\; \sum_{n=1}^{\infty} \frac{8h}{n^2\pi^2} \left[2\sin\frac{n\pi}{4} - \sin\frac{n\pi}{2}\right] \sin\frac{n\pi x}{\ell} \cos\frac{n\pi v t}{\ell}.$$

Problem 4. Solve the previous problem if the initial displacement $y_0(x)$ is described as

$$y_0(x) = \begin{cases} 4hx/\ell, & \text{if } 0 \le x \le \ell/4, \\ 2h - 4hx/\ell, & \text{if } \ell/4 < x \le 3\ell/4, \\ -4h + 4hx/\ell, & \text{if } 3\ell/4 < x \le \ell. \end{cases}$$

Find the displacement as a function of x and t.

Solution. We perform the separation of variables $y(x,t) = X(x)T(t)$ in the same manner as before, use the same boundary conditions to eliminate the cosine part of X, and use the initial conditions to eliminate the sine part of T. Thus,

$$y(x,t) \;=\; \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{\ell} \cos\frac{n\pi v t}{\ell}.$$

The only difference is in the coefficients A_n, which we recalculate as

$$A_n = \frac{2}{\ell}\int_0^\ell y_0(x)\sin\frac{n\pi x}{\ell}\,dx.$$

Note geometrically that $y_0(x) = -y_0(\ell - x)$, i.e. the reflection about $\ell/2$ is precisely the negative of the original curve. Thus,

$$A_n = \int_0^{\ell/2} y_0(x)\sin\frac{n\pi x}{\ell} + y_0(\ell - x)\sin\frac{n\pi(\ell - x)}{\ell}\,dx = \int_0^{\ell/2} y_0(x)(1+\cos n\pi)\sin\frac{n\pi x}{\ell}\,dx.$$

We have used the identity

$$\int_0^{2a} f(t)\,dt = \int_0^a f(t) + f(2a - t)\,dt.$$

This means that $A_{2n+1} = 0$, otherwise, we use our previously calculated value of the integral to obtain

$$A_{2n} = \frac{16h}{(2n)^2\pi^2}\left[2\sin\frac{2n\pi}{4} - \sin\frac{2n\pi}{2}\right] = \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

Thus, we have our solution

$$y(x,t) \;=\; \sum_{n=1}^{\infty}\frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{2n\pi x}{\ell}\cos\frac{2n\pi vt}{\ell}.$$

Problem 5. A string of length ℓ is initially stretched straight; its ends are fixed for all t. At time $t = 0$, its points are given the velocity $V(x) = (\partial y/\partial t)_{t=0}$ described by

$$V(x) = \begin{cases} 2hx/\ell, & \text{if } 0 \le x \le \ell/2,\\ 2h - 2hx/\ell, & \text{if } \ell/2 < x \le \ell.\end{cases}$$

Determine the shape of the string at time t.

Solution. We perform separation of variables $y(x,t) = X(x)T(t)$ again. The boundary conditions dictate that the cosine part of the spatial solution vanishes and $k_n = n\pi x/\ell$. However, since the string is perfectly flat initially, we make no judgement on the temporal part yet. Taking a time derivative, we see that the velocity function dictates $V(0) = V(\ell) = 0$. This means that the derivative of the temporal part, $-C\omega\sin\omega t + D\omega\cos\omega t$, must vanish at 0 and ℓ, so $D = 0$. Thus, our solution is of the form

$$y(x,t) \;=\; \sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{\ell}\sin\frac{n\pi vt}{\ell}.$$

Taking a time derivative,

$$\dot y(x,t) \;=\; \sum_{n=1}^{\infty} A_n\frac{n\pi v}{\ell}\sin\frac{n\pi x}{\ell}\cos\frac{n\pi vt}{\ell}.$$

Now we can set $t = 0$, whereby $\dot y(x,0) = V(x)$ gives us the Fourier coefficients

$$\frac{n\pi v}{\ell}A_n = \frac{2}{\ell}\int_0^\ell V(x)\sin\frac{n\pi x}{\ell}\,dx.$$

If we note that $V(x) = V(\ell - x)$ due to symmetry, this simplifies the integral similarly to the previous problem, so

$$\frac{n\pi v}{\ell}A_n = \frac{2}{\ell}(1-\cos n\pi)\int_0^{\ell/2}\frac{2hx}{\ell}\sin\frac{n\pi x}{\ell}\,dx = \frac{2h}{n^2\pi^2}(1-\cos n\pi)\left[2\sin\frac{n\pi}{2} - n\pi\cos\frac{n\pi}{2}\right].$$

Note that $A_{2n} = 0$, and when n is odd,

$$A_n = \frac{2\ell}{n^3\pi^3 v}(2)\left[2\sin\frac{n\pi}{2}\right] = \frac{8\ell}{n^3\pi^3 v}\sin\frac{n\pi}{2}.$$

This vanishes anyways when n is even. Thus, we have our solution

$$y(x,t) \;=\; \sum_{n=1}^{\infty}\frac{8h\ell}{n^3\pi^3 v}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{\ell}\sin\frac{n\pi vt}{\ell}.$$

Problem 8. Solve Problem 5 if the initial velocity is

$$V(x) = \begin{cases} \sin 2\pi x/\ell, & \text{if } 0 < x < \ell/2, \\ 0, & \text{if } \ell/2 < x < \ell. \end{cases}$$

Solution. We use separation of variables $y(x,t) = X(x)T(t)$ and argue exactly the same as in Problem 5, obtaining the general solution

$$y(x,t) \;=\; \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{\ell} \sin\frac{n\pi vt}{\ell}.$$

To obtain the coefficients A_n, we take a time derivative and set $t = 0$, so

$$\frac{n\pi v}{\ell} A_n = \frac{2}{\ell}\int_0^{\ell} V(x) \sin\frac{n\pi x}{\ell}\,dx = \frac{2}{\ell}\int_0^{\ell/2} \sin\frac{2\pi x}{\ell}\sin\frac{n\pi x}{\ell}\,dx = \frac{4}{(4-n^2)\pi}\sin\frac{n\pi}{2}.$$

Note that when $n = 2$, we must use L'Hôpital's Rule, so

$$A_2 = \frac{\ell}{2\pi v}\lim_{n\to 2}\frac{4}{(4-n^2)\pi}\sin\frac{n\pi}{2} = \frac{\ell}{2\pi v}\lim_{n\to 2}\frac{-1}{n}\cos\frac{n\pi}{2} = \frac{\ell}{4\pi v}.$$

Thus, we have our solution

$$y(x,t) \;=\; \frac{\ell}{4\pi v}\sin\frac{2\pi x}{\ell}\sin\frac{2\pi vt}{\ell} + \sum_{\substack{n=1\\ n\neq 2}}^{\infty} \frac{4\ell}{n(4-n^2)\pi^2 v}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{\ell}\sin\frac{n\pi vt}{\ell}.$$

Problem 12. Let $f(x) = x - x^2$ on $(0,1)$. Expand f as a Fourier sine series and write the corresponding solutions for

- Temperature in a semi-infinite plate with the lower boundary fixed at f.
- Temperature in a rectangular plate of height H with the lower boundary fixed at f.
- Heat flow in one dimension with initial temperature f.
- Particle in a box with initial condition f.
- A plucked string with initial shape f.
- A struck string with initial velocity f.

Solution. We first write f as a Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} A_n \sin n\pi x.$$

The coefficients are calculated as

$$A_n = 2\int_0^1 f(x)\sin n\pi x\,dx = 2\int_0^1 (x - x^2)\sin n\pi x\,dx = \frac{4}{n^3\pi^3}(1-\cos n\pi).$$

We have used the integrals

$$\begin{aligned}
\int_0^1 x\sin n\pi x\,dx &= -\frac{x}{n\pi}\cos n\pi x\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos n\pi x\,dx \\
&= -\frac{1}{n\pi}\cos n\pi. \\
\int_0^1 x^2\sin n\pi x\,dx &= -\frac{x^2}{n\pi}\cos n\pi x\Big|_0^1 + \frac{1}{n\pi}\int_0^1 2x\cos n\pi x\,dx \\
&= -\frac{1}{n\pi}\cos n\pi + \frac{2x}{n^2\pi^2}\sin n\pi x\Big|_0^1 - \frac{2}{n^2\pi^2}\int_0^1 \sin n\pi x\,dx \\
&= -\frac{1}{n\pi}\cos n\pi - \frac{2}{n^3\pi^3}(1-\cos n\pi).
\end{aligned}$$

Thus, A_n vanishes for even n, and is equal to $8/n^3\pi^3$ for odd n. With this, we directly use the equations indicated in the question to write our solutions (setting $\ell = 1$).

(a) Temperature in a semi-infinite plate.

$$T = \sum_{n=1}^{\infty} A_n e^{-n\pi y} \sin n\pi x. \qquad \text{(Eq 2.9)}$$

(b) Temperature in a rectangular plate of height H.

$$T = \sum_{n=1}^{\infty} A_n \left[\sinh n\pi H\right]^{-1} \sinh n\pi(H-y) \sin n\pi x. \qquad \text{(Eq 2.15)}$$

(c) Heat flow in one dimension.

$$u = \sum_{n=1}^{\infty} A_n e^{-(n\pi\alpha)^2 t} \sin n\pi x. \qquad \text{(Eq 3.12)}$$

(d) Particle in a box.

$$\Psi = \sum_{n=1}^{\infty} A_n e^{-iE_n t/\hbar} \sin n\pi x, \qquad E_n = \frac{n^2\pi^2\hbar^2}{2m} \qquad \text{(Eq 3.26)}$$

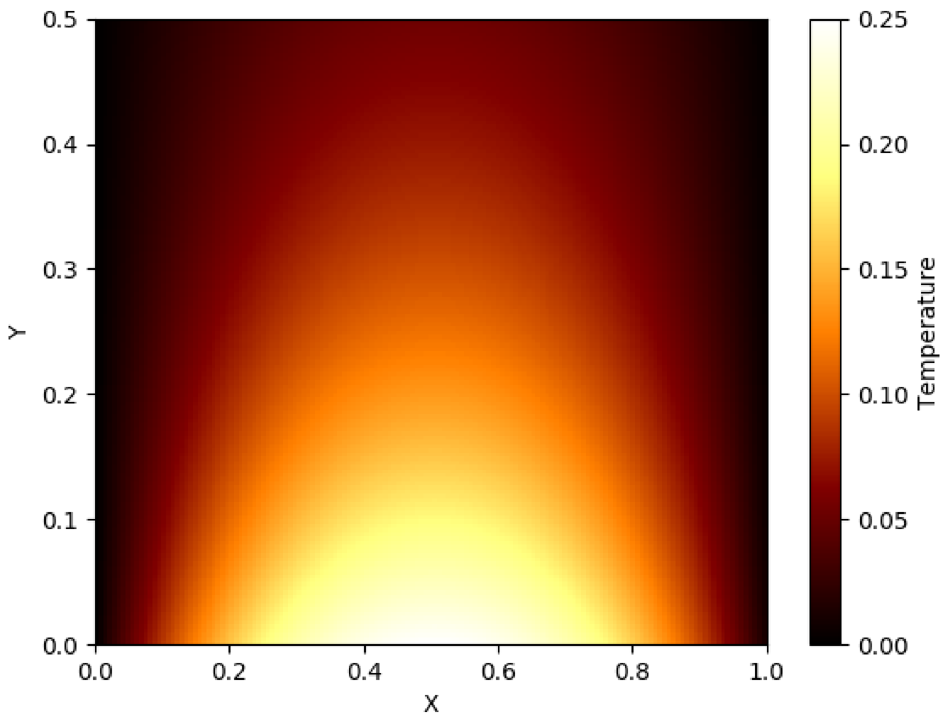
(e) Plucked string.

$$y = \sum_{n=1}^{\infty} A_n \sin n\pi x \cos n\pi v t. \qquad \text{(Eq 4.7)}$$

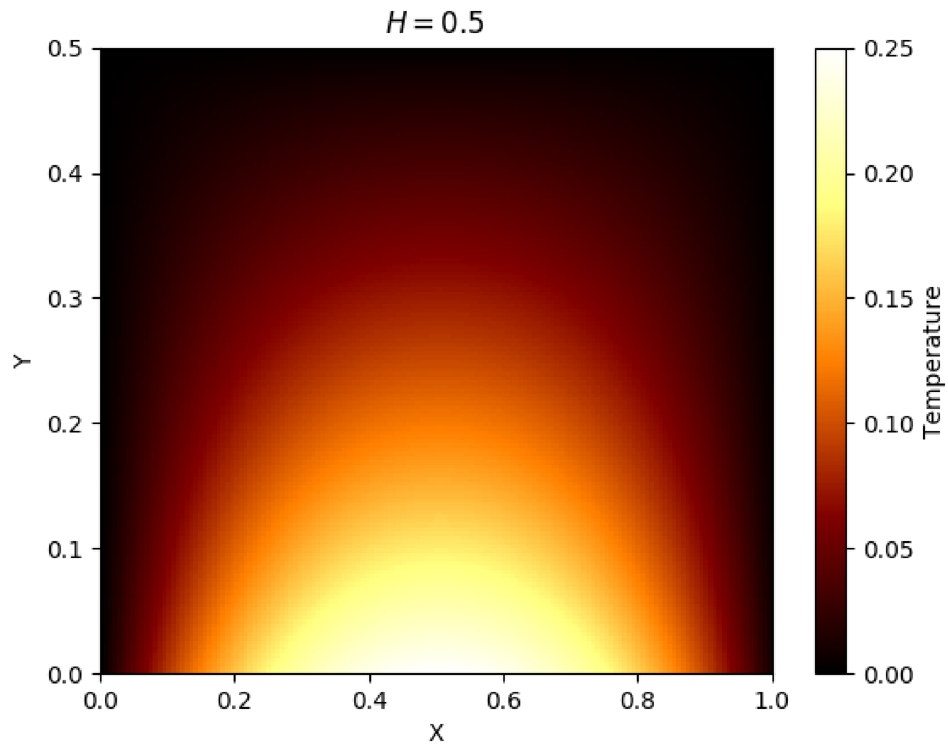
(f) Struck string.

$$y = \sum_{n=1}^{\infty} \frac{A_n}{n\pi v} \sin n\pi x \sin n\pi v t. \qquad \text{(Eq 4.10)}$$

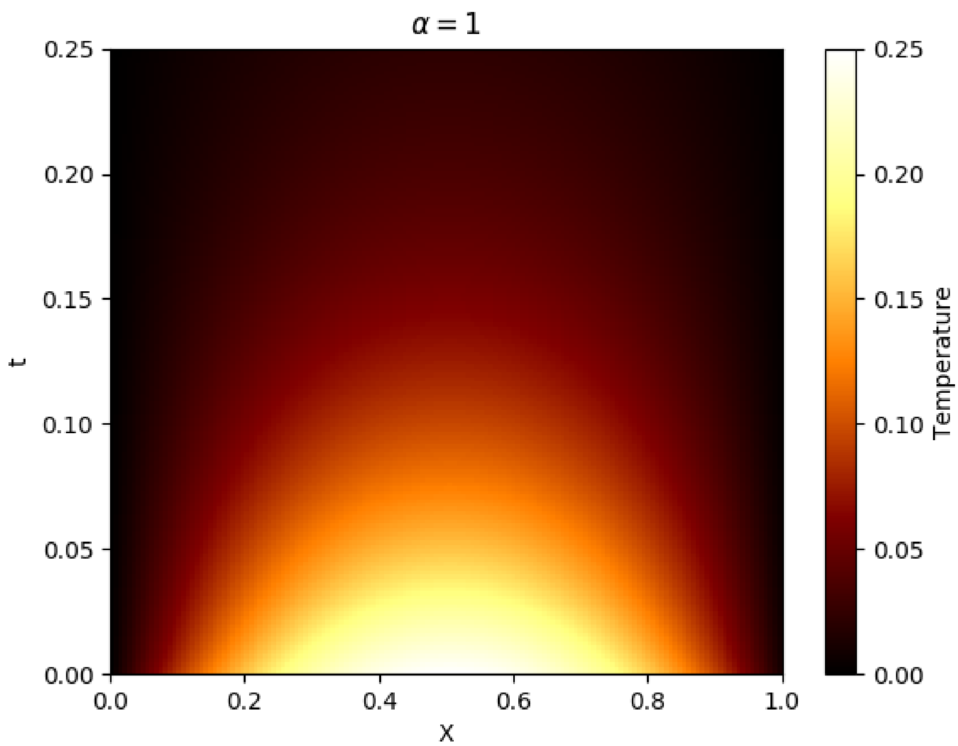
Computer plots of the above solutions are presented below. The code used to generate them can be found here.



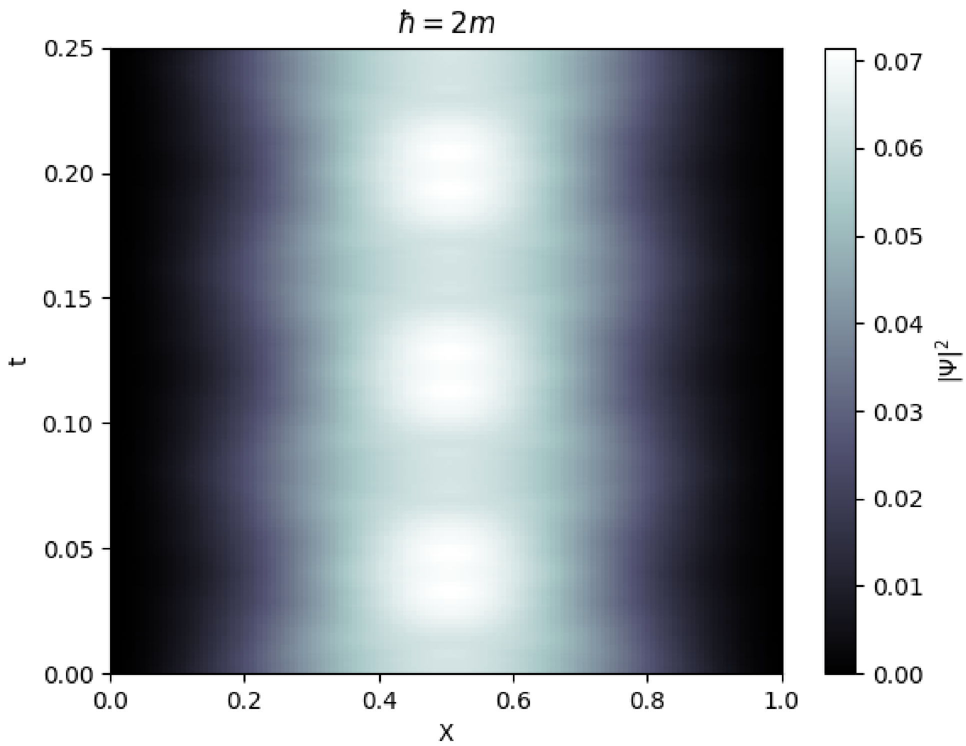
(a) Temperature in a semi-infinite plate



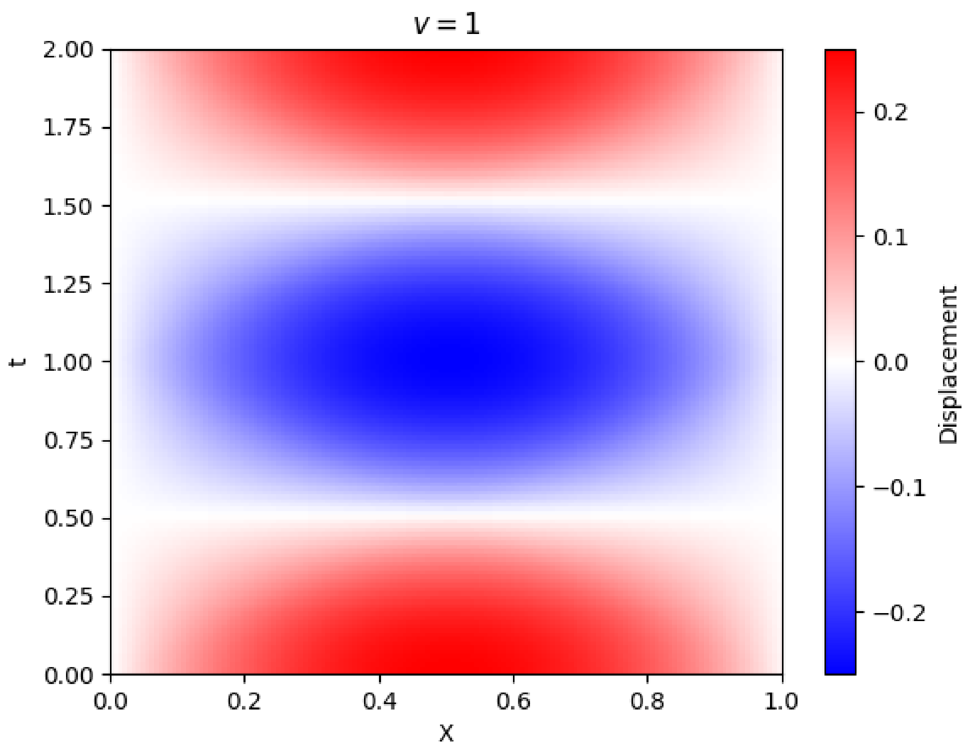
(b) Temperature in a rectangular plate, height H



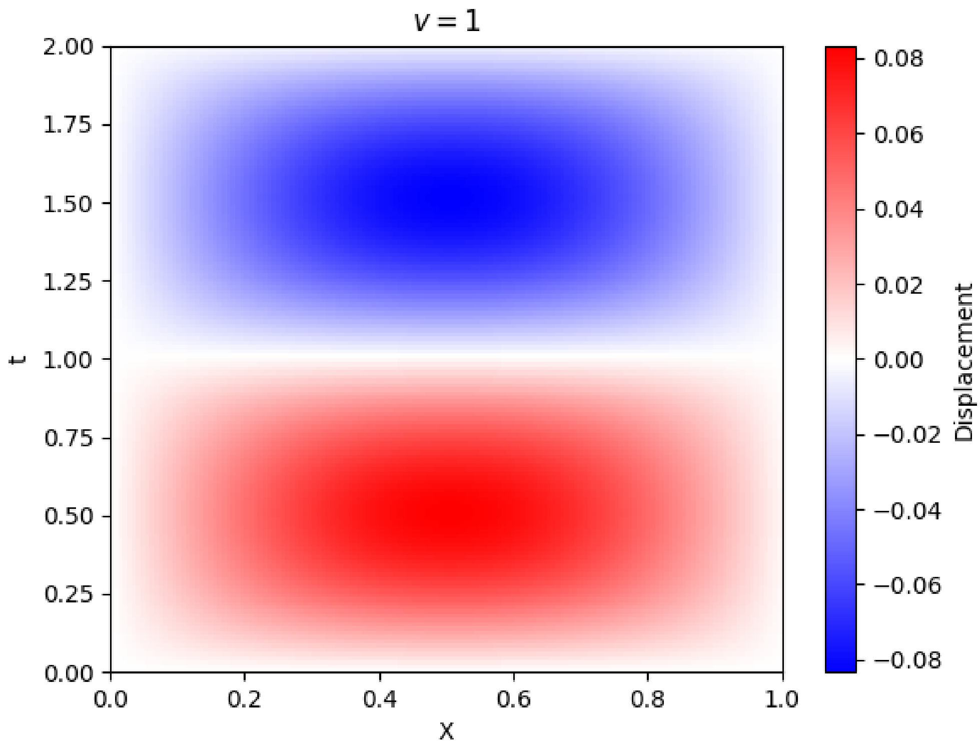
(c) Heat flow in 1D



(d) Particle in a box



(e) Displacement of a plucked string



(f) Displacement of a struck string

Heatmaps of solutions.