MA 2102 : Mathematical Methods II

Satvik Saha, 19MS154 September 11, 2020

Partial Differential Equations (M.L. Boas, Chapter 13)

Section 3. Problem 4. At $t = 0$, two flat slabs each 5 cm thick, one at $0°$ and one at $20°$, are stacked together, and then the surfaces are kept at $0°$. Find the temperature as a function of x and t for $t > 0$.

Solution. We have our initial temperature distribution in the interior

$$u(x, 0) = u_0(x) = \begin{cases} 0, & \text{if } 0 < x < 5 \\ 20, & \text{if } 5 \leq x < 10 \end{cases},$$

and our boundary conditions specify that $u(0, t) = u(10, t) = 0$. With this, we must solve the heat flow equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}.$$

As usual, we perform a partial separation of variables $u(x, t) = F(x)T(t)$, upon which we obtain

$$\frac{1}{F}\frac{d^2}{dx^2}F = \frac{1}{\alpha^2 T}\frac{dT}{dt} = \text{constant} = -k^2.$$

This leads to the eqautions

$$\frac{d}{dt}T +^2 \alpha^2 T = 0, \qquad \frac{d^2}{dx^2}F + k^2 F = 0.$$

The first equation has the solution $T(t) = \exp(-k^2\alpha^2 t)$. Thus, we justify the choice of $-k^2$ as our constant with the observation that T remains finite as $t \to \infty$. The second equation admits solutions of the form $A\cos kx + B\sin kx$. The cosine part is discarded using $u(0, 0) = 0$, and we must have $k = n\pi/10$ from $u(10, 0) = 0$. Superimposing thses solutions, we write

$$u(x, t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{10}\, e^{-(n\pi\alpha/10)^2 t}.$$

The coefficients A_n are simply given by

$$A_n = \frac{2}{10}\int_0^{10} u_0(x)\sin\frac{n\pi x}{10}\,dx = \frac{1}{5}\int_5^{10} 20\sin\frac{n\pi x}{10}\,dx = \frac{40}{n\pi}(\cos n\pi/2 - \cos n\pi).$$

Note that the cosine part in parenthesis is 1 for odd n, 0 for multiples of 4, and -2 for the rest. Thus,

$$u(x, t) = \sum_{n=1}^{\infty} \frac{40}{n\pi}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)\sin\frac{n\pi x}{10}\, e^{-(n\pi\alpha/10)^2 t}.$$

Problem 6. The ends of a bar are initially at $20°$ and $150°$; at $t = 0$, the $150°$ end is changed to $50°$. Find the time-dependent temperature distribution.

Solution. We have already seen that a solution of the heat equation over a domain $[0, \ell]$ where the initial conditions are given by $u_0(x)$ is

$$u(x, t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{\ell} e^{-(n\pi\alpha/\ell)^2 t}.$$

The initial value $u_0(x)$ must be a steady state solution, i.e. $u_0''(x) = 0$. This forces a linear form,

$$u_0(x) = 20 + 130\frac{x}{\ell}.$$

Now, note that at $t \to \infty$, $u \to 0$. However, we want a steady state solution where the boundaries are at 20 and 50 respectively. Such a solution obeys $u_f''(x) = 0$, which is a linear form again.

$$u_f(x) = 20 + 30\frac{x}{\ell}.$$

Thus, our final solution is actually the superposition

$$u(x,t) = u_f(x) + \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{\ell} e^{-(n\pi\alpha/\ell)^2 t}.$$

The coefficients A_n are evaluted at $t = 0$ to obtain

$$A_n = \frac{2}{\ell}\int_0^\ell (u_0(x) - u_f(x)) \sin\frac{n\pi x}{\ell}\, dx = -\frac{200}{n\pi}\cos n\pi.$$

Thus,

$$u(x,t) = 20 + 30\frac{x}{\ell} + \sum_{n=1}^{\infty} \frac{-200}{n\pi}\cos n\pi \sin\frac{n\pi x}{\ell} e^{-(n\pi\alpha/\ell)^2 t}.$$

Note that the coefficients A_n are identical to those in the indicated equation (3.16) because $u_0(x) - u_f(x) = 100x/\ell$, which is the function used in that particular problem.

Problem 7. A bar of length ℓ with insulated sides has its ends also insulated from time $t = 0$ on. Initially the temperature is $u = x$, where x is the distance from one end. Determine the temperature distribution inside the bar at time t.

Solution. We perform separation of variables $u(x,t) = F(x)T(t)$ as usual, obtaining $T(t) = \exp(-k^2\alpha^2 t)$ and $F(x) = A\cos kx + B\sin kx$. Now, since the boundaries are insulated, we demand $u'(0,t) = u'(\ell,t) = 0$, which means that $-A\sin kx + B\cos kx = 0$ at $x = 0, \ell$. This forces $B = 0$ and $k = n\pi/\ell$, so we take superpositions and obtain the general solution

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{\ell} e^{-(n\pi\alpha/\ell)^2 t}.$$

We calculate the coefficients A_n by setting $t = 0$, obtaining a cosine series from which we have

$$A_n = \frac{2}{\ell}\int_0^\ell u_0(x)\cos\frac{n\pi x}{\ell}\, dx.$$

For $n = 0$, we see that $A_0 = \ell$. Otherwise,

$$A_n = \frac{2}{\ell}\int_0^\ell x\cos\frac{n\pi x}{\ell}\, dx = -\frac{2\ell}{n^2\pi^2}(1 - \cos n\pi).$$

Putting everything together,

$$u(x,t) = \frac{\ell}{2} - \sum_{n=1}^{\infty} \frac{2\ell}{n^2\pi^2}(1 - \cos n\pi)\cos\frac{n\pi x}{\ell} e^{-(n\pi\alpha/\ell)^2 t}.$$

Problem 9. A bar of length 2 is initially at 0°. For $t > 0$, the $x = 0$ end is insulated and the $x = 2$ end is held at 100°. Find the time dependent temperature distribution.

Solution. We perform the standard separation of variables $u(x,t) = F(x)T(t)$, obtaining $T(t) = \exp(-k^2\alpha^2 t)$ and $F(x) = A\cos kx + B\sin kx$. Now, from $u'(0,t) = 0$, $F'(0) = B = 0$ and from $u(x \to 2, 0) = 0$, we must have $\cos 2k = 0$, so $2k = (2n+1)\pi/2$. Thus,

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{(2n+1)\pi x}{4} e^{-((2n+1)\pi\alpha/4)^2 t}.$$

This is not quite right, since as $t \to \infty$, we want $u(2, t \to \infty) = 100$ while here, $u(2, t \to \infty) = 0$. We fix this by adding the solution 100 to u, so

$$u(x,t) = 100 + \sum_{n=0}^{\infty} A_n \cos \frac{(2n+1)\pi x}{4} e^{-((2n+1)\pi\alpha/4)^2 t}.$$

Note that we have absorbed the other constant from the cosine series into the sum so that we could take the proper limit as $t \to \infty$. We can now calculate the coefficients

$$A_n = \frac{2}{2} \int_0^2 (-100) \cos \frac{(2n+1)\pi x}{4}\, dx = -\frac{400}{(2n+1)\pi} \cos n\pi.$$

This process of obtaining the Fourier coefficients is justified, since our cosine series conly contains odd terms. Thus, there is no separate calculation for A_0, which would have otherwise been calculated separately if the cosine vanished. Putting everything together,

$$u(x,t) = 100 - \sum_{n=0}^{\infty} \frac{400}{(2n+1)\pi} \cos n\pi \cos \frac{(2n+1)\pi x}{4} e^{-((2n+1)\pi\alpha/4)^2 t}.$$

Problem 12. Solve the "particle" in a box problem to find $\Psi(x,t)$ if $\Psi(x,0) = \sin^2(\pi x)$ on $(0,1)$. What is E_n?

Solution. We start with the Schrödinger equation,

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi(x,t) + V(x)\Psi(x,t) = i\hbar \frac{\partial}{\partial t} \Psi(x,t).$$

Performing a partial separation of variables $\Psi(x,t) = \psi(x)T(t)$, we have

$$-\frac{\hbar^2}{2m} \frac{1}{\psi} \frac{d^2}{dx^2} \psi + V = i\hbar \frac{1}{T} \frac{d}{dt} T = \text{constant } (E).$$

The temporal part is solved by $T(t) = \exp(-iEt/\hbar)$. The spatial part of the equation becomes

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi + V\psi = E\psi.$$

For the "particle in a box" problem, we set $V(x) = 0$ on $(0,1)$ and ∞ everywhere else. This forces $\Psi(x) = 0$ outside $(0,1)$. Setting $k^2 = 2mE/\hbar^2$, we see that a solution of the time independent equation is $\psi(x) = A\cos kx + B\sin kx$. Using the boundary conditions $\Psi(0,t) = \Psi(1,t) = 0$, we must have $A = 0$ and $k = n\pi$. Comparing this with E, we find that

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2m}.$$

Thus, we obtain the general solution

$$\Psi(x,t) = \sum_{n=1}^{\infty} A_n \sin n\pi x \; e^{-iE_n t/\hbar}.$$

The coefficients A_n are calculated by setting $t = 0$, where $\Psi(x,0) = \psi_0(x) = \sin^2 \pi x = (1 - \cos 2\pi x)/2$.

$$A_n = 2\int_0^1 \psi_0(x) \sin n\pi x\, dx = \int_0^1 \sin n\pi x - \cos 2\pi x \sin n\pi x\, dx = \frac{4}{n\pi(4-n^2)}(1 - \cos n\pi).$$

Note that when $n = 2$, we have $A_2 = 0$ (indeed, all the even terms A_{2n} are zero). Putting everything together,

$$\Psi(x,t) = \sum_{n=1}^{\infty} \frac{4}{n\pi(4-n^2)} (1 - \cos n\pi) \sin n\pi x \; e^{-iE_n t/\hbar}.$$

Again, note that the $n = 2$ term vanishes.