MA2102 : Linear Algebra I

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Find a basis of the quotient space $\mathrm{M_n}(\mathbb{R}) \,/\, \mathrm{Sym_n}(\mathbb{R})$, where $\mathrm{Sym_n}(\mathbb{R})$ is the subspace of symmetric matrices.

Solution The subspace of symmetric matrices is such that for any $A \in \mathrm{Sym_n}(\mathbb{R})$,

$$A = A^\top \quad \Leftrightarrow \quad a_{ij} = a_{ji}.$$

Lemma 1. *The skew-symmetric matrices form a subspace where for any $B \in \mathrm{Skew_n}(\mathbb{R})$,*

$$B = -B^\top \quad \Leftrightarrow \quad b_{ij} = -b_{ji}.$$

Proof. Let $B_1, B_2 \in \mathrm{Skew_n}(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Then,

$$(B_1 + B_2)^\top = B_1^\top + B_2^\top = -B_1 - B_2 = -(B_1 + B_2), \qquad (\lambda B_1)^\top = \lambda B_1^\top = -\lambda B_1.$$

Thus, $B_1 + B_2 \in \mathrm{Skew_n}(\mathbb{R})$ and $\lambda B_1 \in \mathrm{Skew_n}(\mathbb{R})$. Furthermore, the zero matrix $\mathbf{0} \in \mathrm{Skew_n}(\mathbb{R})$ because it is equal to its negative transpose. Thus, $\mathrm{Skew_n}(\mathbb{R})$ is a vector subspace of $\mathrm{M_n}(\mathbb{R})$. □

Throughout this proof, we will be using A to denote symmetric matrices and B to denote skew-symmetric matrices.

Lemma 2. *The vector space $\mathrm{M_n}(\mathbb{R})$ can be decomposed as the direct sum of $\mathrm{Sym_n}(\mathbb{R})$ and $\mathrm{Skew_n}(\mathbb{R})$.*

$$\mathrm{M_n}(\mathbb{R}) = \mathrm{Sym_n}(\mathbb{R}) \oplus \mathrm{Skew_n}(\mathbb{R})\,.$$

Equivalently, every matrix $X \in \mathrm{M_n}(\mathbb{R})$ has a unique representation $X = A + B$, where $A \in \mathrm{Sym_n}(\mathbb{R})$ and $B \in \mathrm{Skew_n}(\mathbb{R})$.

Proof. It is easily verified that for any $X \in \mathrm{M_n}(\mathbb{R})$,

$$X = \frac{1}{2}(X + X^\top) + \frac{1}{2}(X - X^\top).$$

Now, $(X + X^\top)/2 \in \mathrm{Sym_n}(\mathbb{R})$ and $(X - X^\top)/2 \in \mathrm{Skew_n}(\mathbb{R})$.

$$\frac{1}{2}(X + X^\top)^\top = \frac{1}{2}(X^\top + X) = \frac{1}{2}(X + X^\top), \qquad \frac{1}{2}(X - X^\top)^\top = \frac{1}{2}(X^\top - X) = -\frac{1}{2}(X - X^\top)$$

Hence, $\mathrm{M_n}(\mathbb{R}) = \mathrm{Sym_n}(\mathbb{R}) + \mathrm{Skew_n}(\mathbb{R})$. Furthermore, $\mathrm{Sym_n}(\mathbb{R}) \cap \mathrm{Skew_n}(\mathbb{R}) = \{\mathbf{0}\}$, because if $X \in \mathrm{Sym_n}(\mathbb{R})$ and $X \in \mathrm{Skew_n}(\mathbb{R})$, we would require $X = X^\top$ and $X = -X^\top$. Equating, we are forced into $X = \mathbf{0}$. This proves the claim. □

Lemma 3. *Let $\beta = \{E_{ij} \colon 1 \leq i, j \leq n\}$ be the standard basis of $\mathrm{M_n}(\mathbb{R})$, where E_{ij} is the matrix with 1 in the i, j^{th} entry and 0 everywhere else. Set $B_{ij} = E_{ij} - E_{ji}$. Then, the set*

$$\gamma = \{B_{ij} \colon 1 \leq i < j \leq n\}$$

is a basis of $\mathrm{Skew_n}(\mathbb{R})$.

Proof. Note that $\gamma \subset \mathrm{Skew_n}(\mathbb{R})$. Consider the following linear combination, where the indices satisfy $1 \leq i < j \leq n$.

$$\sum_{i<j} c_{ij} B_{ij} = \sum_{i<j} c_{ij} E_{ij} - c_{ij} E_{ji} = \mathbf{0}.$$

Note that none of the E_{ij} equals any of the E_{ji} since $i < j$, so the linear independence of β (hence the linear independence of any subset of β) gives $c_{ij} = 0$ for all $1 \leq i < j \leq n$. This establishes the linear

independence of γ.
Suppose $B \in \mathrm{Skew_n}(\mathbb{R})$. Then, $b_{ij} = -b_{ji}$ and $b_{ii} = 0$. This means that we can write

$$\begin{aligned} B = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}E_{ij} &= \sum_{i<j} b_{ij}E_{ij} + \sum_{i=j} b_{ij}E_{ij} + \sum_{i>j} b_{ij}E_{ij} \\ &= \sum_{i<j} b_{ij}E_{ij} + \mathbf{0} + \sum_{i<j} b_{ji}E_{ji} \\ &= \sum_{i<j} b_{ij}E_{ij} - \sum_{i<j} b_{ij}E_{ji} \\ &= \sum_{i<j} b_{ij}B_{ij}. \end{aligned}$$

Thus, γ is a linearly independent spanning subset of $\mathrm{Skew_n}(\mathbb{R})$, hence a basis of $\mathrm{Skew_n}(\mathbb{R})$. □

Let V be vector space over F and let $W \subseteq V$ be a subspace. The quotient space V/W consists of equivalence classes $[\boldsymbol{v}]$, where $\boldsymbol{v} \in V$ and

$$[\boldsymbol{v}] = \boldsymbol{v} + W = \{\boldsymbol{v} + \boldsymbol{w} \colon \boldsymbol{w} \in W\}.$$

Equivalently, $\boldsymbol{u} \in [\boldsymbol{v}]$ if and only if $\boldsymbol{u} - \boldsymbol{v} \in W$. We define

$$[\boldsymbol{u}] + [\boldsymbol{v}] = [\boldsymbol{u} + \boldsymbol{v}], \qquad \lambda[\boldsymbol{v}] = [\lambda \boldsymbol{v}].$$

With this, V/W is a vector space over F.

Lemma 4. *Let $\boldsymbol{v} \in V$ and $\boldsymbol{w}_1, \boldsymbol{w}_2 \in W$. Then*

$$[\boldsymbol{v} + \boldsymbol{w}_1] = [\boldsymbol{v} + \boldsymbol{w}_2].$$

Proof. Pick $\boldsymbol{u} \in [\boldsymbol{v} + \boldsymbol{w}_1]$. Then $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}_1 + \boldsymbol{w}_1'$ for some $\boldsymbol{w}_1' \in W$. Now, $\boldsymbol{w}_1 + \boldsymbol{w}_1' \in W$, so $(\boldsymbol{w}_1 + \boldsymbol{w}_1') - \boldsymbol{w}_2 \in W$. This means that

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}_1 + \boldsymbol{w}_1' = \boldsymbol{v} + \boldsymbol{w}_2 + (\boldsymbol{w}_1 + \boldsymbol{w}_1' - \boldsymbol{w}_2) \in [\boldsymbol{v} + \boldsymbol{w}_2].$$

Hence, $[\boldsymbol{v} + \boldsymbol{w}_1] \subseteq [\boldsymbol{v} + \boldsymbol{w}_2]$. The reverse inclusion follows by symmetry, exchanging the roles of the subscripts 1 and 2 in the argument above.

□

Alternatively, note that since addition in well-defined,

$$[\boldsymbol{v} + \boldsymbol{w}_1] = [\boldsymbol{v}] + [\boldsymbol{w}_1] = [\boldsymbol{v}] + [\mathbf{0}] = [\boldsymbol{v}] + [\boldsymbol{w}_2] = [\boldsymbol{v} + \boldsymbol{w}_2].$$

With this, we claim that the set

$$\gamma' = \{[B_{ij}] \colon B_{ij} \in \gamma\} = \{[B_{ij}] \colon 1 \leq i < j \leq n\}$$

is a basis of $\mathrm{M_n}(\mathbb{R}) \,/\, \mathrm{Sym_n}(\mathbb{R})$.

Consider the linear combination

$$\sum_{i<j} c_{ij}[B_{ij}] \;=\; [\mathbf{0}].$$

By the linearity of the equivalence classes as well as the fact that γ is a basis of $\mathrm{Skew_n}(\mathbb{R})$, we can simplify this as

$$[B] \;=\; [\mathbf{0}],$$

where $B = \sum_{i<j} c_{ij}B_{ij} \in \mathrm{Skew_n}(\mathbb{R})$. This means that the skew-symmetric matrix $B \in [\mathbf{0}]$, i.e. $B = \mathbf{0} + A = A$ for some $A \in \mathrm{Sym_n}(\mathbb{R})$. Lemma 2 forces $B = \mathbf{0}$, whence $c_{ij} = 0$. Thus, γ' is linearly independent.

Pick $[X] \in \mathrm{M_n}(\mathbb{R}) \,/\, \mathrm{Sym_n}(\mathbb{R})$. Since $X \in \mathrm{M_n}(\mathbb{R})$, use Lemma 2 to write $X = A + B$ where $A \in \mathrm{Sym_n}(\mathbb{R})$ and $B \in \mathrm{Skew_n}(\mathbb{R})$. Note that by Lemma 4,

$$[X] = [A + B] = [B],$$

because $A, \mathbf{0} \in \mathrm{Sym_n}(\mathbb{R})$. Now, expand B in the basis γ.

$$B = \sum_{i<j} b_{ij} B_{ij}.$$

Then, the linearity of the equivalence classes gives

$$[X] \;=\; [B] \;=\; \sum_{i<j} b_{ij}[B_{ij}].$$

Thus, γ' is linearly independent and spans $\mathrm{M_n}(\mathbb{R}) \,/\, \mathrm{Sym_n}(\mathbb{R})$. This proves that γ' is a basis of $\mathrm{M_n}(\mathbb{R}) \,/\, \mathrm{Sym_n}(\mathbb{R})$. This completes the exercise.

Moreover, γ and γ' contain $1 + 2 + \cdots + (n-1) = n(n-1)/2$ elements, so

$$\dim \mathrm{Skew_n}(\mathbb{R}) = \dim \mathrm{M_n}(\mathbb{R}) \,/\, \mathrm{Sym_n}(\mathbb{R}) = \frac{1}{2}n(n-1).$$

Remark. *We give a sketch of an alternate proof that γ' is a basis of $\mathrm{M_n}(\mathbb{R}) \,/\, \mathrm{Sym_n}(\mathbb{R})$. Recall that for a linear map $T\colon V \to W$, the map*

$$\mathscr{T} : V/\ker T \to \operatorname{im} T, \qquad [\boldsymbol{v}] \mapsto T(\boldsymbol{v})$$

is a linear isomorphism. By setting

$$T\colon\ \mathrm{M_n}(\mathbb{R}) \to \mathrm{M_n}(\mathbb{R}), \qquad X \mapsto \frac{1}{2}(X - X^\top),$$

note that $\ker T = \mathrm{Sym_n}(\mathbb{R})$ (because $A = A^\top$ when $A \in \mathrm{Sym_n}(\mathbb{R})$) and $\operatorname{im} T = \mathrm{Skew_n}(\mathbb{R})$ (because $(X - X^\top)/2$ is always skew-symmetric).

Note that if $B \in \mathrm{Skew_n}(\mathbb{R})$ then its preimage under $\mathscr{T}$ is the equivalence class $[B]$. Since a linear isomorphism sends a basis to a basis, and the inverse of a linear isomorphism is a linear isomorphism, the set $\mathscr{T}^{-1}(\gamma) = \gamma'$ is a basis of $\mathrm{M_n}(\mathbb{R}) \,/\, \mathrm{Sym_n}(\mathbb{R})$.