# Term presentation
# Problem 6

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# Problem statement

Find a basis of the quotient space $\mathsf{M}_n(\mathbb{R}) \,/\, \mathsf{Sym}_n(\mathbb{R})$, where $\mathsf{Sym}_n(\mathbb{R})$ is the subspace of symmetric matrices.

# Preliminaries

The subspace of symmetric matrices is such that for any $A \in \mathsf{Sym_n}(\mathbb{R})$,

$$A = A^\top \quad \Leftrightarrow \quad a_{ij} = a_{ji}.$$

It can be shown that the skew-symmetric matrices form a subspace such that for any $B \in \mathsf{Skew_n}(\mathbb{R})$,

$$B = -B^\top \quad \Leftrightarrow \quad b_{ij} = -b_{ji}.$$

# Preliminaries

Any matrix $X \in \mathsf{M_n}(\mathbb{R})$ can be written uniquely as the sum of a symmetric and a skew-symmetric matrix.

$$X = \frac{1}{2}(X + X^\top) + \frac{1}{2}(X - X^\top).$$

Furthermore, $\mathsf{Sym_n}(\mathbb{R}) \cap \mathsf{Skew_n}(\mathbb{R}) = \{\mathbf{0}\}$, because if $X$ is both symmetric and skew-symmetric,

$$X^\top = X = -X^\top \quad \Rightarrow \quad X = \mathbf{0}.$$

Thus, we can write

$$\mathsf{M_n}(\mathbb{R}) = \mathsf{Sym_n}(\mathbb{R}) \oplus \mathsf{Skew_n}(\mathbb{R}).$$

# Basis of $\mathsf{Skew_n}(\mathbb{R})$

Let $E_{ij} \in \mathsf{M_n}(\mathbb{R})$ be the matrix whose $i, j^{\text{th}}$ element is 1, and the remaining elements are 0. The set of $\beta$ of all such $E_{ij}$ comprises the standard basis of $\mathsf{M_n}(\mathbb{R})$. For any $X \in \mathsf{M_n}(\mathbb{R})$,

$$X = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} E_{ij}.$$

Set

$$B_{ij} = E_{ij} - E_{ji}.$$

Thus, $B_{ij}$ is a skew-symmetric matrix with 1 in the $i, j^{\text{th}}$ position, and $-1$ in the $j, i^{\text{th}}$ position. Let

$$\gamma = \left\{ B_{ij} \colon 1 \leq i < j \leq n \right\}.$$

# Basis of $\mathsf{Skew_n}(\mathbb{R})$

The linear independence of $\gamma$ follows from the linear independence of $\beta = \{E_{ij}\}$.

$$\sum_{i<j} c_{ij} B_{ij} = \sum_{i<j} c_{ij} E_{ij} - c_{ij} E_{ji} = \mathbf{0}.$$

Suppose $B \in \mathsf{Skew_n}(\mathbb{R})$. Then, $b_{ij} = -b_{ji}$ and $b_{ii} = 0$.

$$\begin{aligned} B = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} E_{ij} &= \sum_{i<j} b_{ij} E_{ij} + \sum_{i=j} b_{ij} E_{ij} + \sum_{i>j} b_{ij} E_{ij} \\ &= \sum_{i<j} b_{ij} E_{ij} + \mathbf{0} + \sum_{i<j} b_{ji} E_{ji} \\ &= \sum_{i<j} b_{ij} E_{ij} - b_{ij} E_{ji} = \sum_{i<j} b_{ij} B_{ij}. \end{aligned}$$

# Quotient spaces

Let $V$ be vector space over $F$ and let $W \subseteq V$ be a subspace. The quotient space $V/W$ consists of equivalence classes $[\boldsymbol{v}]$, where $\boldsymbol{v} \in V$ and

$$[\boldsymbol{v}] = \boldsymbol{v} + W = \{\boldsymbol{v} + \boldsymbol{w} \colon \boldsymbol{w} \in W\}.$$

Equivalently, $\boldsymbol{u} \in [\boldsymbol{v}]$ if and only if $\boldsymbol{u} - \boldsymbol{v} \in W$.

We define

$$[\boldsymbol{u}] + [\boldsymbol{v}] = [\boldsymbol{u} + \boldsymbol{v}], \qquad \lambda[\boldsymbol{v}] = [\lambda \boldsymbol{v}].$$

With this, $V/W$ is a vector space over $F$.

# Quotient spaces

Let $v \in V$ and $w_1, w_2 \in W$. Then

$$[v + w_1] = [v + w_2].$$

Pick $u \in [v + w_1]$. Then $u = v + w_1 + w_1'$ for some $w_1' \in W$. Now, $w_1 + w_1' \in W$, so $(w_1 + w_1') - w_2 \in W$. This means that

$$u = v + w_1 + w_1' = v + w_2 + (w_1 + w_1' - w_2) \in [v + w_2].$$

The reverse inclusion follows by symmetry.

Alternatively, note that since addition in well-defined,

$$[v + w_1] = [v] + [w_1] = [v] + [0] = [v] + [w_2] = [v + w_2].$$

# Basis of $M_n(\mathbb{R}) / \mathrm{Sym}_n(\mathbb{R})$

We claim that the set

$$\gamma' = \{[B_{ij}] \colon B_{ij} \in \gamma\} = \left\{[B_{ij}] \colon 1 \le i < j \le n\right\}$$

is a basis of $\mathsf{M_n}(\mathbb{R}) / \mathsf{Sym_n}(\mathbb{R})$.

Consider the linear combination

$$\begin{aligned} \sum_{i<j} c_{ij}[B_{ij}] &= [\mathbf{0}] \\ [B] &= [\mathbf{0}] \end{aligned}$$

This means that the skew-symmetric matrix $B \in [\mathbf{0}]$, i.e. $B = \mathbf{0} + A = A$ for some $A \in \mathsf{Sym_n}(\mathbb{R})$. This forces $B = \mathbf{0}$, whence $c_{ij} = 0$. Thus, $\gamma'$ is linearly independent.

# Basis of $\mathsf{M}_n(\mathbb{R}) \,/\, \mathsf{Sym}_n(\mathbb{R})$

Pick $[X] \in \mathsf{M}_n(\mathbb{R}) \,/\, \mathsf{Sym}_n(\mathbb{R})$. Since $X \in \mathsf{M}_n(\mathbb{R})$, write $X = A + B$ where $A \in \mathsf{Sym}_n(\mathbb{R})$ and $B \in \mathsf{Skew}_n(\mathbb{R})$. Note that

$$[X] = [A + B] = [B].$$

Now, expand $B$ in the basis $\gamma$.

$$B = \sum_{i<j} b_{ij} B_{ij}.$$

Then,

$$[X] \;=\; [B] \;=\; \sum_{i<j} b_{ij} [B_{ij}].$$

Thus, $\gamma'$ is linearly independent and spans $\mathsf{M_n}(\mathbb{R}) \,/\, \mathsf{Sym_n}(\mathbb{R})$. This proves that $\gamma'$ is a basis of $\mathsf{M_n}(\mathbb{R}) \,/\, \mathsf{Sym_n}(\mathbb{R})$.

Moreover, $\gamma$ and $\gamma'$ contain $1 + 2 + \cdots + (n-1) = n(n-1)/2$ elements, so

$$\dim \mathsf{Skew_n}(\mathbb{R}) = \dim \mathsf{M_n}(\mathbb{R}) \,/\, \mathsf{Sym_n}(\mathbb{R}) = \frac{1}{2}n(n-1).$$

For a linear map $T\colon V \to W$, the map

$$\mathscr{T} : V / \ker T \to \operatorname{im} T, \qquad [v] \mapsto T(v)$$

is a linear isomorphism. By setting

$$T\colon\ \mathrm{M_n}(\mathbb{R}) \to \mathrm{M_n}(\mathbb{R}), \qquad X \mapsto \frac{1}{2}(X - X^\top),$$

note that $\ker T = \mathrm{Sym_n}(\mathbb{R})$ and $\operatorname{im} T = \mathrm{Skew_n}(\mathbb{R})$.

If $T(X) = B \in \mathrm{Skew_n}(\mathbb{R})$, then $X = [B]$. Since a linear isomorphism sends a basis to a basis, and the inverse of a linear isomorphism is a linear isomorphism, the set $\mathscr{T}^{-1}(\gamma) = \gamma'$ is a basis of $\mathrm{M_n}(\mathbb{R}) \,/\, \mathrm{Sym_n}(\mathbb{R})$.