

Term presentation

Problem 5

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Problem statement

Let $a \in \mathbb{R}$. Consider the set

$$S_a^n = \{1, (x - a), (x - a)^2, \dots, (x - a)^n\}.$$

Show that S_a^n is a basis for $P_n(\mathbb{R})$, the space of polynomials of degree at most n .

Any polynomial in $P_n(\mathbb{R})$ is of the form

$$p(x) = a_0 + a_1x + \cdots + a_nx^n.$$

In other words, the set $S_0^n = \{1, x, \dots, x^n\}$ is a basis of $P_n(\mathbb{R})$.
This gives $\dim P_n(\mathbb{R}) = n + 1$.

It is sufficient to show that S_0^n is linearly independent.

The binomial theorem gives

$$(x - a)^n = x^n - nax^{n-1} + \binom{n}{2}a^2x^{n-2} + \dots + (-1)^na^n.$$

Specifically, the coefficient of x^n in $(x - a)^n$ is 1.

This means that

$$(x - a)^n - x^n \in P_{n-1}(\mathbb{R}) \subset P_n(\mathbb{R}).$$

Proof by induction: Base case

For $n = 0$, the claim is trivial. We have $S_a^0 = S_0^0 = \{1\}$, which is a linearly independent set.

For $n = 1$, consider the linear combination of elements from $S_a^1 = \{1, (x - a)\}$

$$c_0 + c_1(x - a) = \mathbf{0},$$

for arbitrary $c_0, c_1 \in \mathbb{R}$.

Successively set $x = a$ and $x = 0$.

Thus, $c_0 = 0$ and $c_0 - c_1a = 0$, whence $c_0 = c_1 = 0$.

Proof by induction: Induction step

Suppose that for $n = k$, the set $S_a^k = \{1, (x - a), \dots, (x - a)^k\}$ is linearly independent.

Consider the linear combination of elements from S_a^{k+1} ,

$$c_0 + c_1(x - a) + \dots + c_k(x - a)^k + c_{k+1}(x - a)^{k+1} = \mathbf{0}.$$

Subtract and add $c_{k+1}x^{k+1}$.

$$\begin{aligned} & \left[c_0 + c_1(x - a) + \dots \right. \\ & \left. + c_k(x - a)^k + c_{k+1} \left((x - a)^{k+1} - x^{k+1} \right) \right] + c_{k+1}x^{k+1} = \mathbf{0}. \end{aligned}$$

Proof by induction: Induction step

The bracketed portion is a polynomial of degree at most k , so we write it in the form

$$p_k(x) = a_0 + a_1x + \cdots + a_kx^k \in P_k(\mathbb{R}).$$

Replacing this in the previous equation,

$$a_0 + a_1x + \cdots + a_kx^k + c_{k+1}x^{k+1} = \mathbf{0}.$$

The linear independence of $S_0^{k+1} = \{1, x, \dots, x^{k+1}\}$ gives $a_0 = a_1 = \cdots = a_k = c_{k+1} = 0$.

Substituting this back into the original linear combination,

$$c_0 + c_1(x - a) + \cdots + c_k(x - a)^k = \mathbf{0}.$$

The induction hypothesis gives $c_0 = c_1 = \cdots = c_k = 0$.

Proof by induction: Conclusion

Thus, by the principle of mathematical induction, the set S_a^n is linearly independent for all integers $n \geq 0$.

Specifically, the set S_a^n is a linearly independent set of size $n + 1$, in the space $P_n(\mathbb{R})$ which has dimension $n + 1$.

Hence, S_a^n is a basis of $P_n(\mathbb{R})$.

Appendix

To show that $\{1, x, x^2, \dots, x^n\}$ is a basis of $P_n(\mathbb{R})$, it suffices to prove its linear independence. Consider the linear combination

$$c_0 + c_1x + c_2x^2 + \cdots + c_nx^n = \mathbf{0}.$$

We choose $n + 1$ distinct reals x , which we exhibit as $n + 1$ roots of the polynomial on the left. However, the degree of this polynomial is at most n .

We conclude that the polynomial on the left is the zero polynomial, so $c_0 = c_1 = \cdots = c_n$.