MA2102 : Linear Algebra I

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Show that a matrix A is of rank 1 if and only if $A = \boldsymbol{x}\boldsymbol{y}^\top$ for some non-zero column vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.

Solution We notate the components of $A \in M_{m\times n}(F)$, $\boldsymbol{x} \in F^m$ and $\boldsymbol{y} \in F^n$ as follows.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \qquad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

Recall that the rank of A is the dimension of the column space of A.

Suppose that $A = \boldsymbol{x}\boldsymbol{y}^\top$ for some non-zero column vectors $\boldsymbol{x} \in F^m$ and $\boldsymbol{y} \in F^n$. We write out the product as

$$\begin{aligned} A = \boldsymbol{x}\boldsymbol{y}^\top &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \\ &= \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_my_1 & x_my_2 & \cdots & x_my_n \end{bmatrix} \\ &= \begin{bmatrix} y_1\boldsymbol{x} & y_2\boldsymbol{x} & \cdots & y_n\boldsymbol{x} \end{bmatrix}. \end{aligned}$$

The column space of A is the span of the columns of A, so for any element $\boldsymbol{v}$ in the column space of A, we can write

$$\boldsymbol{v} = \lambda_1 y_1 \boldsymbol{x} + \lambda_2 y_2 \boldsymbol{x} + \cdots + \lambda_n y_n \boldsymbol{x} = \lambda \boldsymbol{x},$$

for suitable scalars $\lambda_i \in F$. Here, $\lambda = \sum \lambda_i y_i \in F$. Thus, the column space is spanned by the singleton set $\{\boldsymbol{x}\}$. Moreover, $\boldsymbol{x}, \boldsymbol{y} \neq \boldsymbol{0}$ so there is come non-zero component of $\boldsymbol{y}$, say $y_j \neq 0$. Hence, the corresponding column $y_j\boldsymbol{x}$ of A is also non-zero. Thus, the column space contains non-zero elements; specifically, it contains $(y_j\boldsymbol{x})/y_j = \boldsymbol{x}$. Thus, the singleton set $\{\boldsymbol{x}\}$ is linearly independent and spans the column space of A, i.e. is a basis of the column space. This means that $\operatorname{rank} A = 1$.

Suppose that $\operatorname{rank} A = 1$. This means that its column space has dimension 1, i.e. is spanned by a singleton set. Choose a basis $\{\boldsymbol{x}\}$ of the column space, where $\boldsymbol{x} \in F^m$ is non-zero. Because every column of A is in the span of this set, we can write the i^{th} column of A as the linear combination $\lambda_i\boldsymbol{x}$, for suitable scalars $\lambda_i \in F$. Hence,

$$A = \begin{bmatrix} \lambda_1\boldsymbol{x} & \lambda_2\boldsymbol{x} & \cdots & \lambda_n\boldsymbol{x} \end{bmatrix}.$$

Note that all $\lambda_i \in F$ cannot be zero; if this were the case, then A would be the zero matrix, with $\operatorname{rank} A = 0$. Set $\boldsymbol{y} \in F^n$ as the column vector

$$\boldsymbol{y} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}.$$

Observe that $\boldsymbol{y} \neq \boldsymbol{0}$. With this choice of column vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, we have A $= \boldsymbol{x}\boldsymbol{y}^\top$. This completes the proof.