

Term presentation

Problem 4

Satvik Saha, 19MS154

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Problem statement

Show that a matrix A is of rank 1 if and only if $A = \mathbf{xy}^T$ for some non-zero column vectors \mathbf{x} and \mathbf{y} .

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

Preliminaries

The *column rank* of a matrix A is equal to the dimension of the column space of A .

The *row rank* of a matrix A is equal to the dimension of the row space of A .

The column rank and row rank of any matrix $A \in M_{m \times n}(F)$ are equal.

The *rank* of A is equal to the column rank of A .

If $A = \mathbf{xy}^\top$, then $\text{rank } A = 1$

$$\begin{aligned} A = \mathbf{xy}^\top &= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \\ &= \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_m y_1 & x_m y_2 & \cdots & x_m y_n \end{bmatrix} \\ &= \begin{bmatrix} y_1 \mathbf{x} & y_2 \mathbf{x} & \cdots & y_n \mathbf{x} \end{bmatrix}. \end{aligned}$$

If $A = \mathbf{xy}^\top$, then $\text{rank } A = 1$

The column space of A consists of all finite linear combinations of the columns of A .

For any element \mathbf{v} in the column space of A , we can write

$$\mathbf{v} = \lambda_1 y_1 \mathbf{x} + \lambda_2 y_2 \mathbf{x} + \cdots + \lambda_n y_n \mathbf{x} = \lambda \mathbf{x}.$$

The column vector \mathbf{x} spans the column space of A .

Furthermore, there is some $y_i \mathbf{x} \neq \mathbf{0}$ in the column space of A .

$$\text{rank } A \leq 1 \leq \text{rank } A \Rightarrow \text{rank } A = 1.$$

If $\text{rank } A = 1$, then $A = \mathbf{x}\mathbf{y}^\top$

The column space of A admits a singleton basis. Set \mathbf{x} equal to this element.

$$A = \begin{bmatrix} \lambda_1 \mathbf{x} & \lambda_2 \mathbf{x} & \cdots & \lambda_n \mathbf{x} \end{bmatrix}.$$

Set

$$\mathbf{y} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}.$$

With this choice of column vectors \mathbf{x} and \mathbf{y} , we have $A = \mathbf{x}\mathbf{y}^\top$.