MA2102 : Linear Algebra I

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Let V and W be vector spaces over the same field F. Show that the set $\mathcal{L}(V, W)$, consisting of linear maps from V to W, is a vector space. If V and W are finite dimensional, then find the dimension of $\mathcal{L}(V, W)$.

Solution Recall that a vector space V over a field F is a set equipped with a binary operation $+\colon V \times V \to F$ called addition, and an operation $\cdot\colon F \times V \to V$ called scalar multiplication, such that

1. $\boldsymbol{u} + \boldsymbol{v} \in V$, for all $\boldsymbol{u}, \boldsymbol{v} \in V$.
2. $\lambda \boldsymbol{u} \in V$, for all $\boldsymbol{u} \in V$, $\lambda \in F$.
3. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$, for all $\boldsymbol{u}, \boldsymbol{v} \in V$.
4. $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$, for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$.
5. There exists $\mathbf{0} \in V$ such that $\mathbf{0} + \boldsymbol{v} = \boldsymbol{v}$ for all $\boldsymbol{v} \in V$.
6. For all $\boldsymbol{v} \in V$, there exists $\boldsymbol{u} \in V$ such that $\boldsymbol{v} + \boldsymbol{u} = \mathbf{0}$. We denote $\boldsymbol{u} = -\boldsymbol{v}$.
7. $\lambda(\boldsymbol{u} + \boldsymbol{v}) = \lambda \boldsymbol{u} + \lambda \boldsymbol{v}$, for all $\boldsymbol{u}, \boldsymbol{v} \in V$, $\lambda \in F$.
8. $(\lambda\mu)\boldsymbol{v} = \lambda(\mu \boldsymbol{v})$ for all $\boldsymbol{v} \in V$, $\lambda, \mu \in F$.
9. $(\lambda + \mu)\boldsymbol{v} = \lambda \boldsymbol{v} + \mu \boldsymbol{v}$, for all $\boldsymbol{v} \in V$, $\lambda, \mu \in F$.
10. There exists $1 \in F$ such that $1\boldsymbol{v} = \boldsymbol{v}$ for all $\boldsymbol{v} \in V$.

Let $T, T_1, T_2\colon V \to W$ be linear maps and let $\lambda \in F$. We define addition and scalar multiplication on $\mathcal{L}(V, W)$ as follows.

$$\begin{aligned} (T_1 + T_2)(\boldsymbol{v}) &= T_1(\boldsymbol{v}) + T_2(\boldsymbol{v}) && \text{for all } \boldsymbol{v} \in V, \\ (\lambda T)(\boldsymbol{v}) &= \lambda T(\boldsymbol{v}) && \text{for all } \boldsymbol{v} \in V. \end{aligned}$$

With this, we verify the enumerated axioms one by one.

1. Since T_1 and T_2 are linear maps in $\mathcal{L}(V, W)$, for all $\boldsymbol{u}, \boldsymbol{v} \in V$ and $\mu \in F$,

   $$\begin{aligned} (T_1 + T_2)(\boldsymbol{u} + \mu\boldsymbol{v}) &= T_1(\boldsymbol{u} + \mu\boldsymbol{v}) + T_2(\boldsymbol{u} + \mu\boldsymbol{v}) \\ &= T_1(\boldsymbol{u}) + \mu T_1(\boldsymbol{v}) + T_2(\boldsymbol{u}) + \mu T_2(\boldsymbol{v}) \\ &= (T_1 + T_2)(\boldsymbol{u}) + \mu(T_1 + T_2)(\boldsymbol{v}). \end{aligned}$$

   Thus, $T_1 + T_2$ is a linear map in $\mathcal{L}(V, W)$.

2. Again, since T is a linear map in $\mathcal{L}(V, W)$, for all $\boldsymbol{u}, \boldsymbol{v} \in V$ and $\lambda, \mu \in F$,

   $$\begin{aligned} (\lambda T)(\boldsymbol{u} + \mu\boldsymbol{v}) &= \lambda T(\boldsymbol{u} + \mu\boldsymbol{v}) \\ &= \lambda T(\boldsymbol{u}) + \lambda\mu T(\boldsymbol{v})) \\ &= (\lambda T)(\boldsymbol{u}) + \mu(\lambda T)(\boldsymbol{v}). \end{aligned}$$

   Thus, λT is a linear map in $\mathcal{L}(V, W)$.

3. For all $\boldsymbol{v} \in V$, note that the commutativity of addition in W gives

   $$(T_1 + T_2)(\boldsymbol{v}) = T_1(\boldsymbol{v}) + T_2(\boldsymbol{v}) = T_2(\boldsymbol{v}) + T_1(\boldsymbol{v}) = (T_2 + T_1)(\boldsymbol{v}).$$

   Thus, $T_1 + T_2 = T_2 + T_1$.

4. For all $\boldsymbol{v} \in V$, the associativity of addition in W gives

$$((T_1 + T_2) + T_3)(\boldsymbol{v}) = T_1(\boldsymbol{v}) + (T_2 + T_3)(\boldsymbol{v}) = T_1(\boldsymbol{v}) + T_2(\boldsymbol{v}) + T_3(\boldsymbol{v}),$$
$$(T_1 + (T_2 + T_3))(\boldsymbol{v}) = (T_1 + T_2)(\boldsymbol{v}) + T_3(\boldsymbol{v}) = T_1(\boldsymbol{v}) + T_2(\boldsymbol{v}) + T_3(\boldsymbol{v}).$$

   Thus, $(T_1 + T_2) + T_3 = T_1 + (T_2 + T_3)$.

5. Define the linear map $\mathbf{0}_{\mathcal{L}} \colon V \to W$, $\boldsymbol{v} \mapsto \mathbf{0}_W$. For any $T \in \mathcal{L}(V, W)$, for all $\boldsymbol{v} \in V$.

$$(\mathbf{0}_{\mathcal{L}} + T)(\boldsymbol{v}) = \mathbf{0}_{\mathcal{L}}(\boldsymbol{v}) + T(\boldsymbol{v}) = \mathbf{0}_W + T(\boldsymbol{v}) = T(\boldsymbol{v}).$$

   Thus, $\mathbf{0}_{\mathcal{L}} + T = T$.

6. Define $T' \colon V \to W$, $\boldsymbol{v} \mapsto -T(\boldsymbol{v})$. Then for all $\boldsymbol{v} \in V$,

$$(T + T')(\boldsymbol{v}) = T(\boldsymbol{v}) + T'(\boldsymbol{v}) = T(\boldsymbol{v}) - T(\boldsymbol{v}) = \mathbf{0}_W = \mathbf{0}_{\mathcal{L}}(\boldsymbol{v}).$$

   Thus, $T + T' = \mathbf{0}_{\mathcal{L}}$.

7. For $\lambda, \mu \in F$, for all $\boldsymbol{v} \in V$,

$$\begin{aligned}
(\lambda(T_1 + T_2))(\boldsymbol{v}) &= \lambda(T_1 + T_2)(\boldsymbol{v}) \\
&= \lambda(T_1(\boldsymbol{v}) + T_2(\boldsymbol{v})) \\
&= \lambda T_1(\boldsymbol{v}) + \lambda T_2(\boldsymbol{v}) \\
&= (\lambda T_1)(\boldsymbol{v}) + (\lambda T_2)(\boldsymbol{v}) \\
&= (\lambda T_1 + \lambda T_2)(\boldsymbol{v}).
\end{aligned}$$

   Thus, $\lambda(T_1 + T_2) = \lambda T_1 + \lambda T_2$.

8. For $\lambda, \mu \in F$, for all $\boldsymbol{v} \in V$,

$$\begin{aligned}
((\lambda + \mu)T)(\boldsymbol{v}) &= (\lambda + \mu)T(\boldsymbol{v}) \\
&= \lambda T(\boldsymbol{v}) + \mu T(\boldsymbol{v}) \\
&= (\lambda T)(\boldsymbol{v}) + (\mu T)(\boldsymbol{v}) \\
&= (\lambda T + \mu T)(\boldsymbol{v}).
\end{aligned}$$

   Thus, $(\lambda + \mu)T = \lambda T + \mu T$.

9. For $\lambda, \mu \in F$, for all $\boldsymbol{v} \in V$,

$$\begin{aligned}
((\lambda\mu)T)(\boldsymbol{v}) &= (\lambda\mu)T(\boldsymbol{v}) \\
&= \lambda(\mu T(\boldsymbol{v})) \\
&= \lambda(\mu T)(\boldsymbol{v}) \\
&= (\lambda(\mu T))(\boldsymbol{v}).
\end{aligned}$$

   Thus, $(\lambda\mu)T = \lambda(\mu T)$.

10. Pick the scalar $1 \in F$ which satisfies $1\boldsymbol{w} = \boldsymbol{w}$ for all $\boldsymbol{w} \in W$. Then for all $\boldsymbol{v} \in V$,

$$(1T)(\boldsymbol{v}) = 1(T(\boldsymbol{v})) = T(\boldsymbol{v}),$$

   so $1T = T$.

This verifies that $\mathcal{L}(V, W)$ is a vector space.

When V and W are finite-dimensional, say with dimensions n and m respectively, we claim that the dimension of $\mathcal{L}(V, W)$ is mn. To prove this, let $\beta = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be an ordered basis of V and let $\gamma = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$ be an ordered basis of W. Define the linear maps

$$T_{ij} \colon V \to W, \qquad \boldsymbol{v}_k \mapsto \delta_{ik}\boldsymbol{w}_j,$$

for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$. We will show that the set of all such T_{ij} (of which there are mn) comprises a basis of $\mathcal{L}(V, W)$. Note that T_{ij} has only been defined on the basis vectors $\boldsymbol{v}_k \in \beta$. To uniquely define T_{ij} on V, pick arbitrary $\boldsymbol{v} \in V$ and expand it in the basis β as $\boldsymbol{v} = \lambda_1 \boldsymbol{v}_1 + \cdots + \lambda_n \boldsymbol{v}_n$. Note that the choice of scalars $\lambda_i \in F$ is unique, since β is a basis. We now write.

$$T_{ij}(\boldsymbol{v}) = T_{ij}(\lambda_1 \boldsymbol{v}_1 + \cdots + \lambda_n \boldsymbol{v}_n) = \lambda_i \boldsymbol{w}_j.$$

To verify that this map is indeed linear, pick some $\boldsymbol{u} = \mu_1 \boldsymbol{v}_1 + \cdots + \mu_n \boldsymbol{v}_n \in V$. Note that $\boldsymbol{u} + \boldsymbol{v} = (\mu_1 + \lambda_1)\boldsymbol{v}_1 + \ldots (\mu_n + \lambda_n)\boldsymbol{v}_n$, so

$$T_{ij}(\boldsymbol{u} + \boldsymbol{v}) = (\mu_i + \lambda_i)\boldsymbol{w}_j = \mu_i \boldsymbol{w}_j + \lambda_i \boldsymbol{w}_j = T_{ij}(\boldsymbol{u}) + T_{ij}(\boldsymbol{v}).$$

Also, note that $c\boldsymbol{v} = c\lambda_1 \boldsymbol{v}_1 + \cdots + c\lambda_n \boldsymbol{v}_n$ for all $c \in F$, so

$$T_{ij}(c\boldsymbol{v}) = (c\lambda_i)\boldsymbol{w}_j = c(\lambda_i \boldsymbol{w}_j) = cT_{ij}(\boldsymbol{v}).$$

Thus, all T_{ij} are indeed linear maps in $\mathcal{L}(V, W)$.

We first show that the set of T_{ij} spans $\mathcal{L}(V, W)$. Suppose $T \colon V \to W$ is a linear map in $\mathcal{L}(V, W)$. For each of the basis vectors $\boldsymbol{v}_i \in \beta$, there exist unique scalars $a_{ij} \in F$ such that

$$T(\boldsymbol{v}_i) \ = \ a_{i1}\boldsymbol{w}_1 + a_{i2}\boldsymbol{w}_2 + \cdots + a_{im}\boldsymbol{w}_m.$$

This is true because γ is a basis of W. Pick any $\boldsymbol{v} \in V$ and write $\boldsymbol{v} = \lambda_1 \boldsymbol{v}_1 + \cdots + \lambda_n \boldsymbol{v}_n$. Then,

$$T(\boldsymbol{v}) = \sum_{i=1}^{n} \lambda_i T(\boldsymbol{v}_i) = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i a_{ij} \boldsymbol{w}_j = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} T_{ij}(\boldsymbol{v}).$$

This means that

$$T \ = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} T_{ij}.$$

Hence, T is in the span of the set of T_{ij} for arbitrary $T \in \mathcal{L}(V, W)$.

We now show that the set of T_{ij} are linearly independent. Consider the linear combination

$$\sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} T_{ij} = \mathbf{0}_{\mathcal{L}}.$$

By successively evaluating this map on $\boldsymbol{v}_k \in \beta$ for $k = 1, \ldots, n$, we see that

$$\sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} T_{ij}(\boldsymbol{v}_k) = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} \delta_{ik} \boldsymbol{w}_j = \sum_{j=1}^{m} c_{kj} \boldsymbol{w}_j = \mathbf{0}.$$

The linear independence of $\gamma = \{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$ forces $c_{kj} = 0$. Hence, all the coefficients in the linear combination of T_{ij} are zero, so they are linearly independent.

This proves that the set $\{T_{ij} \colon i, j \in \mathbb{N}, 1 \leq i \leq n, 1 \leq j \leq m\}$ is a basis of $\mathcal{L}(V, W)$. This set contains mn elements, hence $\dim \mathcal{L}(V, W) = mn = \dim V \cdot \dim W$.