MA2102 : Linear Algebra I

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A square, $n \times n$ real matrix A is such that $AA^\top$ is diagonal, with each diagonal entry non-zero. Show that the rows of A are orthogonal. Is it true that the columns are orthogonal?

Solution Let $A \in M_n(\mathbb{R})$ be enumerated as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \vdots \\ \boldsymbol{v}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_n \end{bmatrix}.$$

Here, the rows of A are the row vectors $\boldsymbol{v}_i \in \mathbb{R}^n$ and the columns of A are the column vectors $\boldsymbol{w}_i \in \mathbb{R}^n$.

Recall that the standard inner product for row vectors in $\mathbb{R}^n$ is defined as

$$\langle \cdot, \cdot \rangle \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad \langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}\boldsymbol{y}^\top.$$

In other words,

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{n} x_i y_i.$$

We say that the row vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ are orthogonal if $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$.

Lemma 1. *For a row vector $\boldsymbol{x} \in \mathbb{R}^n$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ if and only if $\boldsymbol{x} = \boldsymbol{0}$.*

Proof. Note that the components $x_i \in \mathbb{R}$, so we can write

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle = \sum_{n=1}^{n} x_i x_i = x_1^2 + x_2^2 + \cdots + x_n^2 \geq x_j^2,$$

for all $j = 1, 2, \ldots, n$. If $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$, then we have $0 \geq x_j^2$, which forces $x_j = 0$. Hence, $\boldsymbol{x} = \boldsymbol{0}$.

Conversely, if $\boldsymbol{x} = \boldsymbol{0}$, we clearly see that $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = \sum_i x_i^2 = 0$. □

To relate the elements of the matrix product with the inner products of the rows and columns, we make use of the following lemma.

Lemma 2. *Let $P \in M_{m\times n}(\mathbb{R})$ and $Q \in M_{n\times l}(\mathbb{R})$. Then, the i, j^{th} element of the product PQ is given by*

$$[PQ]_{ij} = \langle \boldsymbol{p}_i, \boldsymbol{q}_j^\top \rangle,$$

where $\boldsymbol{p}_i \in \mathbb{R}^n$ is the i^{th} row of P, and $\boldsymbol{q}_j \in \mathbb{R}^n$ is the j^{th} column of Q.

Proof. This is purely computational, from the definition of matrix multiplication.

$$[PQ]_{ij} = \sum_{k=1}^{n} p_{ik} q_{kj} = \boldsymbol{p}_i \boldsymbol{q}_j = \boldsymbol{p}_i (\boldsymbol{q}_j^\top)^\top = \langle \boldsymbol{p}_i, \boldsymbol{q}_j^\top \rangle.$$ □

With this, we prove the desired statement. Let $AA^\top$ be diagonal, where for non-zero scalars $\lambda_i \in \mathbb{R}$, we have

$$[AA^\top]_{ij} = \lambda_i \delta_{ij}.$$

This means that the i, j^{th} element of the product $AA^\top$ is zero precisely when $i \neq j$. However, note that the j^{th} column of $A^\top$ is the j^{th} row of A. Thus, we have

$$[AA^\top]_{ij} = \boldsymbol{v}_i \boldsymbol{v}_j^\top = \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle.$$

This gives $\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = \lambda_i \delta_{ij}$, i.e. $\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = 0$ if and only if $i \neq j$. In other words, $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ are orthogonal when $i \neq j$ and each $\boldsymbol{v}_i$ is non-zero. Thus, the rows of A are orthogonal.

The columns of A satisfying $AA^\top$ need not be orthogonal. We supply the following counterexample.

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}.$$

Note that the rows are orthogonal because

$$AA^\top = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$$

However, the inner product of the two columns is

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}^\top \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 1(-1) + 2(2) = 3 \neq 0.$$

The standard inner product for two column vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ has been defined as $\boldsymbol{x}^\top \boldsymbol{y} = \sum_i x_i y_i$.