Term presentation
Problem 2

Satvik Saha, 19MS154

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Problem statement

A square, $n \times n$ real matrix A is such that $AA^\top$ is diagonal, with each diagonal entry non-zero. Show that the rows of A are orthogonal. Is it true that the columns are orthogonal?

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_n \end{bmatrix}.$$

The rows of A are the row vectors $\mathbf{v}_i \in \mathbb{R}^n$.
The columns of A are the column vectors $\mathbf{w}_i \in \mathbb{R}^n$.

The standard inner product for row vectors in $\mathbb{R}^n$ is defined as

$$\langle \cdot, \cdot \rangle \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad \langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}\mathbf{y}^\top.$$

In other words,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i.$$

Note that

$$\langle \mathbf{x}, \mathbf{x} \rangle = x_1^2 + x_2^2 + \cdots + x_n^2.$$

Thus, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

Preliminaries

Two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in an inner product space are called orthogonal if $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$.

The matrix product can be concisely expressed in terms of the inner product. Let $P \in M_{m \times n}(\mathbb{R})$ and $Q \in M_{n \times l}(\mathbb{R})$. Then, the i, j^{th} element of the product PQ is given by

$$[PQ]_{ij} = \sum_{k=1}^{n} p_{ik} q_{kj} = \boldsymbol{p}_i \boldsymbol{q}_j = \langle \boldsymbol{p}_i, \boldsymbol{q}_j^\top \rangle.$$

Note that $\boldsymbol{p}_i \in \mathbb{R}^n$ is the i^{th} row of P, and $\boldsymbol{q}_j \in \mathbb{R}^n$ is the j^{th} column of Q.

Proof

Let $AA^\top = D(\lambda_1, \lambda_2, \ldots, \lambda_n)$ for non-zero $\lambda_i \in \mathbb{R}$. Thus, the i, j^{th} element of the product $AA^\top$ is given by

$$[AA^\top]_{ij} = \lambda_i \delta_{ij}.$$

If the row vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathbb{R}^n$ are the rows of A, then the column vectors $\boldsymbol{v}_1^\top, \ldots, \boldsymbol{v}_2^\top \in \mathbb{R}^n$ are the columns of $A^\top$.

$$[AA^\top]_{ij} = \boldsymbol{v}_i \boldsymbol{v}_j^\top = \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle.$$

Thus, $\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = 0$ precisely when $i \neq j$. This proves that the rows $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ of A are orthogonal.

Are the columns of A orthogonal?

No. We supply the following counterexample for $n = 2$.

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}.$$

Note that the rows are orthogonal because

$$AA^\top = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$$

However, the inner product of the two columns is

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}^\top \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 1(-1) + 2(2) = 3 \neq 0.$$

Note that the inner product for two column vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ is defined as $\boldsymbol{x}^\top \boldsymbol{y}$.