MA2102 : Linear Algebra I

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Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}.$$

Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

Solution Recall that an eigenvalue of a matrix $A \in M_n(\mathbb{R})$ is $\lambda \in \mathbb{R}$ such that $A\boldsymbol{v} = \lambda\boldsymbol{v}$, for some non-zero $\boldsymbol{v} \in \mathbb{R}^n$. Furthermore, the eigenvalues of a matrix can be computed as the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I_n)$.

Lemma 1. *The eigenvalues of $A \in M_n(F)$ are precisely the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I_n)$.*

Proof. Suppose that λ is an eigenvalue of A. Then, we find some $\boldsymbol{v} \neq \boldsymbol{0}$ such that $A\boldsymbol{v} = \lambda\boldsymbol{v}$. Rearranging, $(A - \lambda I_n)\boldsymbol{v} = \boldsymbol{0}$. Thus, $\boldsymbol{v} \in \ker(A - \lambda I_n)$, so the kernel of $A - \lambda I_n$ must have dimension greater than 1. From the Rank Nullity theorem, $A - \lambda I_n$ cannot be of full rank, hence $\det(A - \lambda I_n) = 0$.

Conversely, note that if $\det(A - \lambda I_n) = 0$, then $A - \lambda I_n$ is not of full rank. Thus, the Rank Nullity theorem guarantees that we can find some $\boldsymbol{v} \neq \boldsymbol{0}$ in the kernel of $A - \lambda I_n$ such that $(A - \lambda I_n)\boldsymbol{v} = \boldsymbol{0}$, i.e. $A\boldsymbol{v} = \lambda\boldsymbol{v}$. This shows that λ is an eigenvalue of A, completing the proof. □

Lemma 2. *Let $P \in M_n(F)$ be a matrix whose columns are eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ of A. Then, $AP = PD$, where D is the diagonal matrix of the corresponding eigenvalues.*

Proof. Suppose that $A\boldsymbol{v}_i = \lambda_i\boldsymbol{v}_i$, for $\lambda_i \in F$. The rules of matrix multiplication directly give the desired result.

$$\begin{aligned} AP &= \begin{bmatrix} A\boldsymbol{v}_1 & A\boldsymbol{v}_2 & \cdots & A\boldsymbol{v}_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1\boldsymbol{v}_1 & \lambda_2\boldsymbol{v}_2 & \cdots & \lambda_n\boldsymbol{v}_n \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \\ &= PD. \end{aligned}$$

□

Suppose that we find n linearly independent eigenvalues of A. Then from Lemma 2, $AP = PD$ where the columns of P are the eigenvectors. Furthermore, the linear independence of the eigenvectors mean that P is of full rank, i.e. is invertible. Thus, we can write $P^{-1}AP = D$.

The remainder of this exercise is purely computational. We first evaluate the characteristic polynomial

$$\begin{aligned} \det \begin{bmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{bmatrix} &= (1-\lambda)^3 + 2 \cdot 8 - 3 \cdot 4(1-\lambda) \\ &= 1 - 3\lambda + 3\lambda^2 - \lambda^3 + 16 - 12 + 12\lambda \\ &= 5 + 9\lambda + 3\lambda^2 - \lambda^3. \end{aligned}$$

The Rational Root theorem says that any rational root of p must be of the form ± 1 or ± 5. Upon inspection, $p(-1) = p(5) = 0$. We synthetically divide $p(\lambda)$ by $\lambda - 5$.

$$\begin{array}{r|rrrr} \lambda - 5 & -1 & 3 & 9 & 5 \\ & 0 & -5 & -10 & -5 \\ \hline & -1 & -2 & -1 & 0 \end{array}$$

Thus,

$$p(\lambda) = (\lambda - 5)(-\lambda^2 - 2\lambda - 1) = (5 - \lambda)(1 + \lambda)^2.$$

The only roots of $p(\lambda)$ are 5 and -1. From Lemma 1, the eigenvalues of A are 5 and -1. We calculate the associated eigenvectors by demanding $(A - \lambda_i I_n) = \boldsymbol{v}_i$.

When $\lambda = 5$,

$$\begin{bmatrix} 1-5 & 2 & 2 \\ 2 & 1-5 & 2 \\ 2 & 2 & 1-5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -4v_1 + 2v_2 + 2v_3 \\ 2v_1 - 4v_2 + 2v_3 \\ 2v_1 + 2v_2 - 4v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This forces $v_1 = v_2 = v_3$. We choose $v_1 = v_2 = v_3 = 1$.

When $\lambda = -1$,

$$\begin{bmatrix} 1+1 & 2 & 2 \\ 2 & 1+1 & 2 \\ 2 & 2 & 1+1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2v_1 + 2v_2 + 2v_3 \\ 2v_1 + 2v_2 + 2v_3 \\ 2v_1 + 2v_2 + 2v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives $v_1 + v_2 + v_3 = 0$. We choose two vectors such that $v_2 = 0, v_3 = 1$ in the first and $v_2 = 1, v_3 = 0$ in the second.

Thus, we have obtained three eigenvectors of A, chosen to be linearly independent.

$$\lambda = 5, \qquad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda = -1, \qquad \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \boldsymbol{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Note that this is not the complete set of eigenvectors associated with each of the eigenvalues. The eigenspace E_5 is the set $\operatorname{span}\{\boldsymbol{v}_1\} = \{c_1\boldsymbol{v}_1 \colon c_1 \in \mathbb{R}\}$ and the eigenspace E_{-1} is the set $\operatorname{span}\{\boldsymbol{v}_2, \boldsymbol{v}_3\} = \{c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 \colon c_2, c_3 \in \mathbb{R}\}$.

The linear independence of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ is verified by Gauss Jordan elimination. Let P be the matrix whose columns are the eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_3, \boldsymbol{v}_3$ in order.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Perform the row operation $R_1 \to R_1 + R_2 + R_3$.

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Perform $R_2 \to R_2 - \frac{1}{3}R_1$ and $R_3 \to R_3 - \frac{1}{3}R_1$.

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{array}\right]$$

Perform $R_1 \to \frac{1}{3}R_1$, and the swap $R_2 \leftrightarrow R_3$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{array}\right]$$

We have the identity matrix I_3 on the left, which shows that P is full rank. Thus, P is invertible, and its inverse is the matrix on the right. We have already proved that we must have $P^{-1}AP = D$, where D is the diagonal matrix of eigenvalues $D(5, -1, -1)$. Nevertheless, we verify this computationally, completing the exercise.

$$\begin{aligned}
P^{-1}AP &= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \\
&= \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 5 & 1 & 1 \\ 5 & 0 & -1 \\ 5 & -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}
\end{aligned}$$