MA 2101 : Analysis I

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Exercise 1 Show that the sum, difference, product and quotient (if the denominator is non-zero everywhere in the domain of definition) of two real valued continuous functions on a metric space M are continuous.

Exercise 2 Let (X, d) be a metric space and let $f, g\colon C \to \mathbb{R}$ be continuous. Show that the set

$$A = \{x \in X \colon f(x) < g(x)\}$$

is an open set in X.

Solution Note that the difference of continuous function $g - f$ is continuous, and the set $(0, \infty)$ is open in $\mathbb{R}$. Thus, the preimage of the open set $(0, \infty)$ in $g - f$, i.e. the set $\{x \in X \colon 0 < (g - f)(x)\}$, must be open in X. This is precisely the set A, which proves the claim.

Exercise 3 Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences and let $N \in \mathbb{N}$ be such that $\alpha_n \leq \beta_n$ for all $n \geq N$. Show that

$$\liminf_{n\to\infty} \alpha_n \leq \liminf_{n\to\infty} \beta_n.$$

Exercise 4 For $n \in \mathbb{N}$, define $\alpha_n = n^{1/n}$. Does the sequence $\{\alpha_n\}$ converge? Determine

$$\limsup_{n\to\infty} \alpha_n.$$

Solution We show that the sequence $n^{1/n} \to 1$. Note that for $n \geq 2$, we have $n^{1/n} > 1$, so we write $n^{1/n} = 1 + h_n$ for positive h_n. Thus, using the binomial theorem,

$$n = (1 + h_n)^n = 1 + nh_n + \frac{1}{2}n(n-1)h_n^2 + \cdots + h_n^n > \frac{1}{2}n(n-1)h_n^2.$$

Thus, $0 < h_n^2 < 2/(n-1)$, which means that $h_n \to 0$ by the Comparison Test, so $n^{1/n} = 1 + h_n \to 1$. Since $n^{1/n} \neq 0$, we also see that $1/n^{1/n} \to 1$.

Now, if $\alpha_n \to \ell$, then the set of subsequential limits is the singleton set $\{\ell\}$. To show this, suppose that some subsequence $\alpha_{n_k} \to \ell'$, where $\ell' \neq \ell$. Set $\epsilon = |\ell - \ell'|/3 > 0$. Then, there exists some $N_1 \in \mathbb{N}$ such that $\alpha_{n_k} \in B_\epsilon(\ell')$ for all $n_k \geq N_1$. Also, since $\alpha_n \to \ell$, there exists some $N_2 \in \mathbb{N}$ such that $\alpha_n \in B_\epsilon(\ell)$ for all $n \geq N_2$. Now, choose $N > N_1 + N_2$, and choose K such that $n_K > N$. Then, $\alpha_{n_K} \in B_\epsilon(\ell)$ and $\alpha_{n_K} \in B_\epsilon(\ell')$. Now, from the triangle inequality,

$$0 < 3\epsilon = |\ell - \ell'| = |(\ell - \alpha_{n_K}) - (\ell' - \alpha_{n_K})| \leq |\ell - \alpha_{n_K}| + |\ell' - \alpha_{n_K}| < \epsilon + \epsilon = 2\epsilon.$$

This is a contradiction. Thus, we must have $\ell = \ell'$. This forces the set of subsequential limits to be the singleton $\{\ell\}$, whose supremum is ℓ. Thus,

$$\limsup_{n\to\infty} n^{1/n} = 1.$$

Exercise 5

(a) If $\{a_n\}$ is a decreasing sequence of strictly positive numbers and if $\sum a_n$ is convergent, show that $\lim_{n\to\infty} na_n = 0$.

(b) Give an example of a divergent series $\sum a_n$ with $\{a_n\}$ decreasing and such that $\lim_{n\to\infty} na_n = 0$.

Solution

(a) Since $\sum a_n$ converges and $\{a_n\}$ is decreasing, we use the Cauchy Condensation test to conclude that $\sum 2^n a_{2^n}$ converges. Thus, the sequence of the terms $2^n a_{2^n} \to 0$. Let $\epsilon > 0$, and let $N \in \mathbb{N}$ be such that for all $n \geq N$, $0 < 2^n a_{2^n} < \epsilon/2$. (recall that a_n is strictly positive). Thus, for all $n \geq 2^N > N$, find $k \geq N$ such that $2^{k+1} > n \geq 2^k$. Then, $a_{2^{k+1}} < a_n \leq a_{2^k}$, so

$$0 < na_n \leq na_{2^k} < 2^{k+1}a_{2^k} < \epsilon.$$

Thus, we have $na_n \to 0$.

We have used the fact that $2^n > n$ for all $n \in \mathbb{N}$. This is shown by induction. The base case $n = 1$ is true; if $2^k > k$ for some $k \in \mathbb{N}$, note that $k \geq 1$ so $2^{k+1} = 2 \cdot 2^k > 2k = k + k \geq k + 1$. This completes the proof.

(b) Starting the sequence from $n = 2$, set

$$a_n = \frac{1}{n \log n}.$$

Note that $\log(n+1) > \log(n)$, so $\{a_n\}$ is strictly decreasing. Also, $\lim_{n\to\infty} na_n = \lim_{n\to\infty} 1/\log n = 0$, because $\log n$ is strictly positive for $n > 1$ and strictly increasing, so $1/\log n$ is a strictly positive decreasing sequence bounded below by 0. Hence, $1/\log n$ converges via the Monotone Convergence theorem. Also, the subsequence $1/\log 2^n = 1/n \log 2$ converges to 0, hence $1/\log n \to 0$.

Applying the Cauchy Condensation test,

$$2^n a_{2^n} = \frac{2^n}{2^n \log 2^n} = \frac{1}{n \log 2}.$$

We know that $\sum 1/n$ does not converge. Thus, neither does $\sum 2^n a_{2^n}$, so $\sum a_n$ diverges.

Exercise 6

(a) Let

$$s_k = \sum_{n=0}^{k} \frac{1}{n!}.$$

Show that $\sum_{k=0}^{\infty} k(e - s_k)$ converges.

(b) Show that $e \notin \mathbb{Q}$.

Solution

(a) We note that $e = \sum_{n=1}^{\infty} 1/n!$, so

$$\begin{aligned}
e - s_k = \sum_{n=k+1}^{\infty} \frac{1}{n!} &= \frac{1}{(k+1)!} + \frac{1}{(k+2)!} + \dots \\
&= \frac{1}{k!}\left[\frac{1}{k+1} + \frac{1}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} + \dots\right] \\
&\leq \frac{1}{k!}\left[\frac{1}{k+1} + \frac{1}{(k+1)^2} + \frac{1}{(k+1)^3} + \dots\right] \\
&= \frac{1}{k!} \cdot \frac{1}{1-(k+1)} \\
&= \frac{1}{k \cdot k!}.
\end{aligned}$$

We used a comparison with a geometric series. Also, each term in $e - s_k$ is positive. Thus, we obtain the inequality

$$0 < k(e - s_k) \leq \frac{1}{k!}.$$

The series $\sum 1/k!$ converges (to e), so the given series $\sum k(e - s_k)$ must also converge by the comparison test.

(b) Suppose to the contrary that $e \in \mathbb{Q}$, so $e = p/q$ for some integers $p, q > 0$. Note that $q \neq 1$, since e is not an integer (it can be shown that $2 < e < 3$). Now, $q!/n!$ is an integer whenever $q \geq n$, because $q! = q \cdot (q-1)! = \cdots = q \cdot (q-1) \cdots (n+1) \cdot n!$. Thus, the quantity

$$q! s_q = \sum_{n=0}^{q} \frac{q!}{n!}$$

is the sum of positive integers, hence is a positive integer. Also, $q!e = q! \cdot p/q = (q-1)! \cdot p$ is an integer, hence $q!(e - s_k)$ is an integer. However, we have already shown that

$$0 < q!(e - s_q) \leq \frac{1}{q} < 1,$$

which is a contradiction since there are no integers in $(0,1)$. Hence, $e \notin \mathbb{Q}$.

Exercise 7 For $x \in \mathbb{R}$, let

$$E_x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

(a) Show that the series E_x converges for all $x \in \mathbb{R}$.

(b) Show that if $|x| < 2$, then $(E_x - 2)(2 - |x|) \leq |x|$.

Solution

(a) Let $a_n = x^n/n!$. Note that

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \left| \frac{x}{n+1} \right| \to 0 < 1.$$

Thus, by the ratio test, the series E_x converges everywhere, for all $x \in \mathbb{R}$.

(b) We have to show that if $|x| < 2$,

$$E_x \leq 2 + \frac{|x|}{2 - |x|} = 2 + \frac{|x|/2}{1 - |x|/2}.$$

Equivalently,

$$E_x - 1 \leq 1 + \frac{|x|/2}{1 - |x|/2} = \frac{1}{1 - |x|/2}.$$

Expanding as a power series and a geometric series respectively, we have to show that

$$x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots \leq 1 + \frac{|x|}{2} + \frac{|x|^2}{4} + \frac{|x|^3}{8} + \cdots + \frac{|x|^n}{2^n} + \ldots$$

Note that the series of absolute values is greater than the original by comparison, so we have to show that

$$|x| + \frac{|x|^2}{2} + \frac{|x|^3}{6} + \cdots + \frac{|x|^n}{n!} + \cdots \leq 1 + \frac{|x|}{2} + \frac{|x|^2}{4} + \frac{|x|^3}{8} + \cdots + \frac{|x|^n}{2^n} + \ldots$$

Whenever $n \geq 8$, we see that $n! > 2^n$, hence $|x|^n/n! < |x|^n/2^n$. Thus, we need only show that

$$|x| + \frac{|x|^2}{2} + \frac{|x|^3}{6} + \frac{|x|^4}{24} + \frac{|x|^5}{120} + \frac{|x|^6}{720} + \frac{|x|^7}{5040} \leq 1 + \frac{|x|}{2} + \frac{|x|^2}{4} + \frac{|x|^3}{8} + \frac{|x|^4}{16} + \frac{|x|^5}{32} + \frac{|x|^6}{64} + \frac{|x|^7}{128}.$$

Set $u = |x| \geq 0$, rearrange, and note that we demand the following when $0 \leq u < 2$.

$$1 - \frac{u}{2} - \frac{u^2}{4} - \frac{u^3}{24} + \frac{u^4}{48} + \frac{11u^5}{480} + \frac{41u^6}{2880} + \frac{307u^7}{40320} \geq 0.$$

Clearing the common denominator 40320, we write

$$40320 + 840u^4 + 924u^5 + 574u^6 + 307u^7 \geq 20160u + 10080u^2 + 1680u^3.$$

We have used the fact that $n! > 2^n$ for $n \geq 8$. Thus is true by induction. The base case $n = 8$ is $8! > 2^8 = 256$. Suppose that $k! \geq 2^k$ for some $k \in \mathbb{N}$; then $(k+1)! = k(k+1)! \geq k \cdot 2^k > 2^{k+1}$.

Exercise 8 Justify whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$$

converges absolutely or diverges.

Solution The given series converges conditionally. Note that the series $\sum(-1)^n$ has partial sums -1 and 0, hence the partial sums are bounded. Also, the sequence $1/\sqrt{n^2+1}$ is decreasing and converges to zero, since

$$0 < \frac{1}{\sqrt{n^2+1}} < \frac{1}{\sqrt{n^2}} < \frac{1}{n},$$

so $1/\sqrt{n^2+1} \to 0$. Thus, by Abel's Lemma, the series $\sum_{n=1}^{\infty}(-1)^n/\sqrt{n^2+1}$ converges.

The series of absolute values is such that

$$0 < \frac{1}{2n} = \frac{1}{\sqrt{n^2+3n^2}} < \frac{1}{\sqrt{n^2+1}}.$$

Note that the harmonic series $\sum 1/n$ diverges. Thus, the series $\sum_{n=1}^{\infty} 1/\sqrt{n^2+1}$ diverges by the comparison test.