MA 2101 : Analysis I

Satvik Saha, 19MS154, Group C November 9, 2020

Exercise 1 Let $\{\alpha_n\}$ be a sequence of positive integers such that the digit '5' occurs in the decimal expansion of none of the terms of this sequence. Show that the series $\sum_{n=1}^{\infty} 1/\alpha_n$ converges.

Solution We assume that α_n are distinct. Without loss of generality[1], let $\{\alpha_n\}$ be an increasing sequence. Let A_m denote the set of all such α_n such that $10^m \leq \alpha_n < 10^{m+1}$. Every integer in A_m has precisely $m+1$ digits. The first cannot be 0 or 5, and the remaining m digits cannot be 5. Thus, A_m can contain no more than 8×9^m integers. Furthermore, the smallest element of A_m has to be at least 10^m, so $1/\alpha_n \leq 1/10^m$ for all $\alpha_n \in A_m$. Thus,

$$\sum_{\alpha_n \in A_m} \frac{1}{\alpha_n} \leq |A_m| \frac{1}{10^m} \leq \frac{8 \cdot 9^m}{10^m}.$$

Also, note that all A_m, with $m = 0, 1, 2, \dots$, exhaust all possible terms of $\{\alpha_n\}$ exclusively. Thus, if s_k denotes the partial sum of all terms $1/\alpha_n$ where $\alpha_n < 10^k$,

$$s_k = \sum_{\alpha_n < 10^k} \frac{1}{\alpha_n} \leq \sum_{m=0}^{k-1} 8 \cdot \left(\frac{9}{10}\right)^m = 8 \cdot \frac{1 - (9/10)^k}{1 - 9/10} < 80.$$

Thus, the partial sums are bounded above by 80. Furthermore, the sequence of partial sums is monotonically increasing, since all the terms in the series are positive. Thus, the given series $\sum_{n=1}^{\infty} 1/\alpha_n$ converges by the Monotone Convergence Theorem.

Exercise 2 Let $k \in \mathbb{Z}$. Find the radius of convergence of the of the power series $\sum_{n=1}^{\infty} x^n/n^k$.

Solution To find the radius of convergence of the given power series, we must calculate the limit

$$a = \limsup_{n\to\infty} \left|\frac{1}{n^k}\right|^{1/n}.$$

When $k = 0$, this limit is trivially 1.
We first show that the sequence $n^{1/n} \to 1$. Note that for $n \geq 2$, we have $n^{1/n} > 1$, so we write $n^{1/n} = 1 + h_n$ for positive h_n. Thus, using the binomial theorem,

$$n = (1 + h_n)^n = 1 + nh_n + \frac{1}{2}n(n-1)h_n^2 + \dots + h_n^n > \frac{1}{2}n(n-1)h_n^2.$$

Thus, $0 < h_n^2 < 2/(n-1)$, which means that $h_n \to 0$, so $n^{1/n} = 1 + h_n \to 1$. Since $n^{1/n} \neq 0$, we also see that $1/n^{1/n} \to 1$. Thus, taking k (or $-k$) products, we see that $1/n^{k/n} \to 1$, so in all cases, $a = 1$. Thus, the radius of convergence of the power series is $1/a = 1$ irrespective of k.

Note that we can also see this via the ratio test, whereby as $n \to \infty$,

$$\left|\frac{x^{n+1}}{(n+1)^k} \cdot \frac{n^k}{x^n}\right| = |x| \left|\frac{n}{n+1}\right|^k \to |x|.$$

Thus, the series converges when $|x| < 1$.

Exercise 3 Let $\{\alpha_n\}$ be a real sequence. Show that if $\sum_{n=1}^{\infty} n\alpha_n$ converges, then so does the series $\sum_{n=1}^{\infty} \alpha_n$.

Solution We simply apply Abel's Lemma on the sequences $\{n\alpha_n\}$ and $\{1/n\}$. The first converges as given, hence its k^{th} partial sums $s_k = \sum_{n=1}^{k} \alpha_n$ are bounded[2]. Also, the sequence $1/n \to 0$, since for

[1] Since we prove convergence, thereby absolute convergence of the increasing sequence, any rearrangement is also convergent by Dirichlet's Theorem.

[2] We may define $\alpha_0 = 0$ for consistency with the definitions.

any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $N\epsilon > 1$. Thus, $1/n \in B_\epsilon(0)$ for all $n \geq N$. Furthermore, $1/(n+1) < 1/n$, so this sequence is non-increasing. Thus, the series formed by their product,

$$\sum_{n=1}^{\infty} n\alpha_n \cdot \frac{1}{n} = \sum_{n=1}^{\infty} \alpha_n,$$

must also converge.

Exercise 4 Let $\{\alpha_n\}$ be a real sequence. Show that

(a) If the ratio test implies that the series $\sum_{n=1}^{\infty} \alpha_n$ converges, then so does the root test.

(b) If the root test implies that the series $\sum_{n=1}^{\infty} \alpha_n$ diverges, then so does the ratio test.

Solution

(a) Suppose the ratio test gives the convergence of $\sum_{n=1}^{\infty} \alpha_n$, i.e.

$$\limsup_{n\to\infty} \left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \ell < 1.$$

Thus, given $\epsilon > 0$, there exists an integer $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\left|\frac{\alpha_{n+1}}{\alpha_n}\right| < \ell + \epsilon.$$

We thus telescope the product

$$|\alpha_n| = \left|\frac{\alpha_n}{\alpha_{n-1}}\right| \dots \left|\frac{\alpha_{N+1}}{\alpha_N}\right| |\alpha_N| < (\ell + \epsilon)^{n-N} |\alpha_N|.$$

Thus, for all $n \geq N$, we have

$$|\alpha_n|^{1/n} < (\ell+\epsilon)^{1-N/n} |\alpha_N|^{1/n} = (\ell+\epsilon) \left|\frac{\alpha_N}{(\ell+\epsilon)^N}\right|^{1/n}.$$

Taking the limit $n \to \infty$, we have

$$\limsup_{n\to\infty} |\alpha_n|^{1/n} \leq \ell + \epsilon.$$

Simply choosing $\epsilon = (1-\ell)/2$, we have $\limsup_{n\to\infty} |\alpha_n|^{1/n} < 1$, as desired.

Note that we have not verified whether the proper limit exists, merely the fact that the upper limit is less than 1.

We have used the fact that for $a > 0$, the limit $a^{1/n} \to 1$. To prove this, first suppose $a > 1$, in which case $a^{1/n} > 1$. We thus write $a^{1/n} = 1 + b_n$, so for $n \geq 2$,

$$a = (1+b_n)^n = 1 + nb_n + \dots + b_n^n > nb_n.$$

Thus, $0 < b_n < a/n$, so $b_n \to 0$, hence $a^{1/n} \to 1$. For $a < 1$, simply note that $1/a > 1$, and $(1/a)^{1/n} \to 1$, so $a^{1/n} \to 1$. The case $a = 1$ is trivial.

(b) Suppose the root test gives the divergence of $\sum_{n=1}^{\infty} \alpha_n$, i.e.

$$\limsup_{n\to\infty} |\alpha_n|^{1/n} = \ell^* > 1.$$

Thus, given $\epsilon > 0$, there exists a subsequence $\alpha_{k_n} \to \ell$ such that $1 < \ell \leq \ell^*$[3], and[4]

$$\ell - \epsilon \leq |\alpha_{k_n}|^{1/k_n} \leq \ell + \epsilon.$$

[3] Note that ℓ^* is the supremum of subsequential limits.

[4] Simply choose k_1 as the first index where the sequence is contained within the ϵ neighbourhood.

This means that $(\ell - \epsilon)^{k_n} \leq |\alpha_{k_n}| \leq (\ell + \epsilon)^{k_n}$. Specifically, since $\ell > 1$, we can always choose ϵ such that $\ell - \epsilon > 1$. For instance, we may set $\epsilon = (\ell - 1)/4$.

Suppose that $\limsup_{n\to\infty} |\alpha_{n+1}/\alpha_n| = s \leq 1$. Note that $s \geq 0$. This means that for the same ϵ, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

$$\left|\frac{\alpha_{n+1}}{\alpha_n}\right| < s + \epsilon.$$

Thus, for all $k_n > N$, we can telescope the product

$$|\alpha_{k_n}| = \left|\frac{\alpha_{k_n}}{\alpha_{k_n-1}}\right| \cdots \left|\frac{\alpha_{N+1}}{\alpha_N}\right| |\alpha_N| < (s+\epsilon)^{k_n-N}|\alpha_N|.$$

Since $(\ell - \epsilon)^{k_n} < |\alpha_{k_n}|$, this is equivalent to demanding

$$\left(\frac{\ell - \epsilon}{s+\epsilon}\right)^{k_n} < \frac{|\alpha_N|}{(s+\epsilon)^N}.$$

On the other hand, note that $(\ell - \epsilon)/(s+\epsilon) > 1$, since with our choice of $\epsilon = (\ell - 1)/4$,

$$\ell - \epsilon = \ell - 4\epsilon + 3\epsilon = 1 + 3\epsilon \geq s + 3\epsilon > s + \epsilon.$$

Thus, the quantity $((\ell - \epsilon)/(s+\epsilon))^{k_n}$ is unbounded above with increasing $k_n > N$. This is a contradiction. Thus, we must have $s > 1$, as desired.