MA 2101 : Analysis I

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Exercise 1 Show that if $\{\alpha_n\}$ is a Cauchy sequence in the Euclidean space $\mathbb{R}$, then so is $\{\alpha_n^2\}$. Does the converse also hold?.

Solution Since $\{\alpha_n\}$ is a Cauchy sequence in $\mathbb{R}$, it converges to a real number L. We claim that $\alpha_n^2 \to L^2$, which in turn means that $\{\alpha_n^2\}$ is a Cauchy sequence[1].

Let $\epsilon > 0$ be arbitrary. Note that the convergent sequence $\{\alpha_n\}$ must be bounded, i.e. $|\alpha_n| < M$ for some positive $M \in \mathbb{R}$. This means that for all $n \in \mathbb{N}$,

$$|\alpha_n + L| \leq |\alpha_n| + |L| < M + |L|.$$

Furthermore, we can choose $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|\alpha_n - L| < \frac{\epsilon}{M + |L|}.$$

Thus, for all $n \geq N$, we have

$$|\alpha_n^2 - L^2| = |\alpha_n - L||\alpha_n + L| < (M + |L|)\frac{\epsilon}{M + |L|} = \epsilon.$$

This establishes that $\alpha_n^2 \to L^2$.

The converse is not true. Consider the sequence defined by $\alpha_n = (-1)^n$. Clearly, the constant sequence $\{\alpha_n^2\} = \{1\}$ converges to 1 and hence is Cauchy, but the sequence $\{\alpha_n\}$ does not converge and hence is not Cauchy.

Exercise 2 Let (M, d) be a metric space and let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in M such that

$$\lim_{n\to\infty} d(\alpha_n, \beta_n) = 0.$$

Prove the following statements.

(a) The sequence $\{\alpha_n\}$ is convergent if and only if the sequence $\{\beta_n\}$ is convergent.

(b) The sequence $\{\alpha_n\}$ is a Cauchy sequence if and only if $\{\beta_n\}$ is a Cauchy sequence.

Solution

(a) Suppose $\alpha_n \to p$, where $p \in M$. This means that for all reals $r > 0$, there exists $N \in \mathbb{N}$ such that $\alpha_n \in B_r(p)$ for all $n \geq N$. We claim that $\beta_n \to p$.

Let $r > 0$ be arbitrary, and let $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $\alpha_n \in B_{r/2}(p)$. Because $d(\alpha_n, \beta_n) \to 0$, we can choose $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $d(\alpha_n, \beta_n) < r/2$. Thus, setting $N = N_1 + N_2$, we observe that for all $n \geq N$, the triangle inequality gives

$$d(\beta_n, p) \leq d(\beta_n, \alpha_n) + d(\alpha_n, p) < \frac{r}{2} + \frac{r}{2} = r.$$

Thus, $\beta_n \in B_r(p)$ for all $n \geq N$, which proves that the sequence $\{\beta_n\}$ converges.

The converse follows trivially by swapping the roles of $\{\alpha_n\}$ and $\{\beta_n\}$.

[1] Cauchy sequences and convergent sequences are precisely the same in the Euclidean space $\mathbb{R}$.

(b) Suppose $\{\alpha_n\}$ is a Cauchy sequence. This means that for all reals $r > 0$, there exists $N \in \mathbb{N}$ such that $d(\alpha_m, \alpha_n) < r$ when both $m, n \geq N$.

Let $r > 0$ be arbitrary, and let N_1 be such that for all $m, n \geq N_1$, $d(\alpha_m, \alpha_n) < r/3$. Because $d(\alpha_n, \beta_n) \to 0$, we can choose $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $d(\alpha_n, \beta_n) < r/3$. Thus, whenever both $m, n \geq N_1 + N_2$, repeated application of the triangle inequality gives

$$d(\beta_m, \beta_n) \leq d(\beta_m, \alpha_m) + d(\alpha_m, \alpha_n) + d(\alpha_n, \beta_n) < \frac{r}{3} + \frac{r}{3} + \frac{r}{3} = r.$$

This shows that $\{\beta_n\}$ is Cauchy.

Again, the converse follows trivially by swapping the roles of $\{\alpha_n\}$ and $\{\beta_n\}$.

Exercise 3 Let $\{\alpha_n\}$ be a sequence of positive real numbers such that

$$|\alpha_{n+2} - \alpha_{n+1}| < |\alpha_{n+1} - \alpha_n|,$$

for all $n \in \mathbb{N}$. Can we conclude from the above condition that the sequence $\{\alpha_n\}$ converges?

Solution No. Consider the sequence defined by $\alpha_n = \sqrt{n}$. We see that

$$\sqrt{n+2} - \sqrt{n+1} = \frac{1}{\sqrt{n+2} + \sqrt{n+1}} < \frac{1}{\sqrt{n} + \sqrt{n+1}} = \sqrt{n+1} - \sqrt{n}.$$

Thus, the sequence $\{\alpha_n\}$ satisfies the given condition. However, the sequence is clearly unbounded in the reals, and hence cannot converge.

Another counterexample is the sequence defined by

$$\alpha_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Clearly,

$$|\alpha_{n+2} - \alpha_{n+1}| = \frac{1}{n+2} < \frac{1}{n+1} = |\alpha_{n+1} - \alpha_n|.$$

On the other hand, the harmonic series diverges, so the sequence $\{\alpha_n\}$ does not converge.