MA 2101 : Analysis I

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Exercise 1 Using only the definition of an ordered field, show that if $x, y \in \mathbb{R}$ with $x > y > 0$, then

$$\sqrt{x} > \sqrt{y}.$$

Solution Note that $\sqrt{x} > 0$ and $\sqrt{y} > 0$, since if either were equal to 0, then one of x and y would be zero. Also, $\sqrt{x} \neq \sqrt{y}$ since if it were, then $x = \sqrt{x}\sqrt{x} = \sqrt{y}\sqrt{y} = y$. Suppose that $\sqrt{x} < \sqrt{y}$. Then, $\sqrt{y} - \sqrt{x} > 0$ and $\sqrt{y} + \sqrt{x} > 0$, so

$$(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x}) \;=\; y - x > 0,$$

which is a contradiction. Thus, we must have $\sqrt{x} > \sqrt{y}$.

Exercise 2 Show that $\mathbb{Q}$ is neither open nor closed in the Euclidean space $\mathbb{R}$.

Solution We know that between two reals, there exists a rational real as well as an irrational real[1]. Thus, if $p \in \mathbb{Q}$ were an interior point of $\mathbb{Q}$, there would exist $r > 0$ such that $B_r(q) \subseteq \mathbb{Q}$. This is impossible since there exists an irrational number $x \notin \mathbb{Q}$ between p and $p+r$. Thus, $p \notin \mathbb{Q}^\circ$, so $\mathbb{Q}$ is not open.

Again, if $\mathbb{Q}$ were closed, then $S = \mathbb{R} \setminus \mathbb{Q}$ must be open. If $x \in S$ were an interior point of S, then there would exist $r > 0$ such that $B_r(x) \subseteq S$. This is also impossible, since there exists a rational number $y \notin S$ between x and $x + r$. Thus, $p \notin S^\circ$, so S is not open, hence $\mathbb{Q}$ is not closed.

Exercise 3 Find the closure of the set

$$S = \left\{ \frac{1}{n^2} : n \in \mathbb{N} \right\},$$

in the Euclidean space $\mathbb{R}$.

Solution We claim that $\overline{S} = S \cup \{0\}$. Since all points in S are trivially closure points of S, we first show that 0 is a ilmit point of S. This follows since for every $r > 0$, we can find $n \in \mathbb{N}$ such that $nr > 1$ using the Archimedean property. Since, $n^2 \geq n$, we have $0 < 1/n^2 < r$, so $1/n^2 \in S \cap B_r(0)$.

We now show that there are no limit points of S apart from 0. Suppose $x \notin S \cup \{0\}$ is a limit point of S. This means that every neighbourhood of x contains infinitely many points of S. If $x < 0$, then note that $B_{-x}(x) \cap S = \emptyset$. Otherwise, if $x > 0$, set $r = x/2$. If $B_r(x)$ contained infinitely many points of S, then there would be infinitely many natural numbers n such that $x - r < 1/n^2 < x + r$, i.e. inifintely many n such that $n^2 < 2/x$, which is absurd. Hence, x is not a limit point of S.

Exercise 4 Construct an example where an infinite union of compact subsets of the Euclidean space $\mathbb{R}$ is a bounded open set S. Is S compact?

[1] This is true because given $x, y \in \mathbb{R}$, $x < y$, we can choose rationals p, q such that $x < p < y$ and $x < p < q < y$, using the density of the rationals in the reals. We also pick a rational s such that $p - \sqrt{2} < s < q - \sqrt{2}$, and it is easily verified that $s + \sqrt{2}$ is irrational.

Solution Compact subsets of $\mathbb{R}$ are precisely the closed and bounded sets, by the Heine-Borel theorem. Thus, consider

$$\mathcal{O} = \bigcup_{n=1}^{\infty} \mathcal{C}_n, \qquad C_n = \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right].$$

Note that each $\mathcal{C}_n$ is a closed interval contained within the open ball $(-1, 1)$, hence is compact. We claim that $\mathcal{O} = (-1, 1)$. This is true because for any $x \in \mathcal{O}$, we find n such that $x \in C_n$, so $-1 + 1/n \leq x \leq 1 + 1/n$. Thus, $-1 < x < 1$, i.e. $x \in (-1, 1)$. Again, for any $x \in (-1, 1)$, we find $m, n \in \mathbb{N}$ such that $m(x+1) > 1$, and $n(1-x) > 1$, so $-1+1/m < x < 1-1/n$. Setting $k = \max(m, n)$, we see that $x \in \mathcal{C}_k$, hence $x \in \mathcal{O}$. Additionally, $\mathcal{O}$ is bounded and open, as it is simply the open ball $B_1(0)$.

It follows from the Heine-Borel theorem that the set $\mathcal{O}$ is not compact, since it is not closed. Note that $1 \notin \mathcal{O}$ is a limit point of $\mathcal{O}$, since for any $r > 0$, we see that $1 - r/2 \in \mathcal{O} \cap B_r(1)$ if $r < 4$ and $0 \in \mathcal{O} \cap B_r(1)$ if $r \geq 4$.

Exercise 5

(i) Show that the map $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$d(x, y) = \frac{\sqrt{\|x - y\|}}{1 + \sqrt{\|x - y\|}},$$

is a metric, where $\|\|$ denotes the Euclidean norm on $\mathbb{R}^n$.

(ii) Let d be the above metric. Show that not all closed and bounded subsets of $(\mathbb{R}^n, d)$ are compact.

(iii) Does the above phenomenon provide a counterexample to the Heine-Borel theorem?

Solution

(i) The fact that d is symmetric, i.e. $d(x, y) = d(y, x)$, follows trivially from the fact that the Euclidean norm $\|\|$ is symmetric.

$$d(x, y) = \frac{\sqrt{\|x - y\|}}{1 + \sqrt{\|x - y\|}} = \frac{\sqrt{\|y - x\|}}{1 + \sqrt{\|y - x\|}} = d(y, x).$$

The non-negativity of the Euclidean norm guarantees that $\|x - y\| \geq 0$, so $\sqrt{\|x - y\|} \geq 0$ and $1 + \sqrt{\|x - y\|} > 0$. This makes $d(x, y)$ well defined and non-negative for all $x, y \in \mathbb{R}^n$. Moreover, if $d(x, y) = 0$, the denominator is positive so the numerator $\sqrt{\|x - y\|}$ must be zero. This forces $\|x - y\| = 0$, whence $x = y$. Again, if $x = y$, then $\|x - y\| = 0$ so $d(x, y) = 0$.

We must now show that d obeys the triangle inequality. Set[2] $a^2 = \|x-y\|$, $b^2 = \|y-z\|$, $c^2 = \|x-z\|$. Thus, from the properties of the Euclidean norm,

$$c^2 \leq a^2 + b^2.$$

Note that a, b, c are non-negative reals. Thus, the following set of inequalities are bidirectionally equivalent.

$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) \\ \frac{c}{1 + c} &\leq \frac{a}{1 + a} + \frac{b}{1 + b} \\ c(1 + a)(1 + b) &\leq a(1 + b)(1 + c) + b(1 + a)(1 + c) \\ c + ac + bc + abc &\leq a + ab + ac + abc + b + ab + bc + abc \\ c &\leq a + b + 2ab + abc \end{aligned}$$

The last inequality is true since $(a + b)^2 = a^2 + b^2 + 2ab \geq a^2 + b^2$, so

$$c \leq \sqrt{a^2 + b^2} \leq a + b \leq a + b + 2ab + abc.$$

This proves the desired inequality.

[2] This is justified, since $\sqrt{\|x\|} \geq 0$.

(ii) We claim that the set $\mathbb{R}^n \subset (\mathbb{R}^n, d)$ is closed, bounded, and not compact. The fact that $\mathbb{R}^n$ is closed follows from the fact that its complement $\emptyset$ is open. The fact that it is bounded follows from the fact that $\mathbb{R}^n \subseteq B_1(0)$. This is true since for any $x \in \mathbb{R}^n$, $\|x\| \geq 0$, so

$$d(x,0) = \frac{\sqrt{\|x\|}}{1+\sqrt{\|x\|}} = 1 - \frac{1}{1+\sqrt{\|x\|}} < 1.$$

Note that if $\|x\| < \|y\|$, then $\sqrt{\|x\|} < \sqrt{\|y\|}$, so $1/(1+\sqrt{\|x\|}) > 1/(1+\sqrt{\|y\|})$, hence $d(x,0) < d(y,0)$.

Let us use the notation $n' = (n, 0, 0, \ldots, 0) \in \mathbb{R}^n$. To show that $\mathbb{R}^n$ is not compact, consider the open cover

$$\mathcal{O} = \bigcup_{n=1}^{\infty} \mathcal{O}_n, \qquad \mathcal{O}_n = B_{d(n',0)}(0).$$

Note that in the this is indeed an open cover of $\mathbb{R}^n$, since for any $x \in \mathbb{R}^n$, we can find a positive integer k such that $\|x\| < \|k'\| = k$ using the Archimedean property. This means that $d(x,0) < d(k',0)$, so $x \in \mathcal{O}_k(0)$. On the other hand, if $\mathcal{O}$ had a finite subcover, note that $\mathcal{O}_n(0) \subset \mathcal{O}_{n+1}(0)$, so our subcover is simply $\mathcal{O}_k(0)$ for some $k \geq 1$. However, we see that $\|k' + 1'\| > \|k'\|$, so $d(k'+1', 0) > d(k', 0)$ and $k' + 1' \notin \mathcal{O}_k(0)$ which is a contradiction. Hence, $\mathbb{R}^n$ is not compact for any $n \in \mathbb{N}$.

(iii) The Heine-Borel theorem applies only to Euclidean spaces $\mathbb{R}^n$, with the Euclidean metric. Since our example in (ii) works in a different metric d, there is no violation of the Heine-Borel theorem.